Axiomatising Ex-lattices

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Abstract

In the 2023 paper 'A fundamental non-classical logic', W. Holliday introduced from a proof-theoretic perspective a *fundamental logic*, which is a common generalisation of both orthologic (a weakening of quantum logic) and intuitionistic logic. In fact, Holliday proved that his set of Fitch-style natural deduction rules leads to *the* weakest introduction-elimination logic. Algebraically, fundamental logic corresponds to bounded lattices $\mathbf{L}$ with a *weak pseudocomplementation* $^{-}$, i.e., the pair $(^{-}, ^{-})$ forms a Galois connection on $\mathbf{L}$ and $\mathbf{L} \models x \cdot \overline{x} \approx 0$, where multiplication denotes conjunction (meet).

Such structures are named *fundamental lattices* by Aguilera and Massas who recently gave an answer to the problem posed by Holliday of describing the intersection of orthologic and intuitionistic logic. They extend the axiomatisation of fundamental lattices by the complicated axiom (Ex)

$$\overline{x(yw+yu)} \cdot x(w+v) \cdot \overline{\overline{z}} \leq \overline{\overline{xz}} \cdot (xw+xv+z) \cdot (y(w+u) + \overline{y(w+u)})$$

with universally quantified variables, where + denotes disjunction, juxtaposition denotes multiplication (conjunction), binding stronger than disjunction, and the vinculum represents negation. Aguilera and Massas name the subclass of fundamental lattices satisfying the axiom (Ex) *Ex-lattices* and prove that this class of structures can also be axiomatised by the axioms of fundamental lattices and three laws using at most four variables. They ask, however, about the three-variable fragment of Ex-logic.

We will show that Ex-lattices form a variety that can be axiomatised by the bounded lattice equations and seven additional identities (or universal inequalities) each mentioning at most three variables. All but one of these seven are common identities of ortholattices and pseudocomplemented lattices, that is, we can pinpoint the influence of distributivity coming from the theory of Heyting lattices, i.e., from intuitionistic logic.

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