Steiner forms

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Basic definitions

- Let F be a field and let V be a finite-dimensional vector space over F. A trilinear form $f : V^3 \to F$ is alternating if

$$f(u_1, u_2, u_3) = 0 \text{ whenever } u_i = u_j \text{ for } i \neq j.$$

- A (trilinear alternating) form f is equivalent to g ($f \sim g$) if there exists $\phi \in \mathbf{GL}(V)$ such that

$$f(u_1, u_2, u_3) = g(\phi(u_1), \phi(u_2), \phi(u_3)).$$

- Classification of classes of $\sim$ is known up to dimension 7 for a large family of fields including all finite fields.
- A vector u belongs to the radical of f (denoted by $\mathrm{Rad} f$) if

$$f(u_1, u_2, u_3) = 0 \text{ whenever } u = u_i \text{ for some i.}$$

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Radical polynomial

- Radical of a vector $u \in V$ is the subspace $\mathrm{Rad}(u) = \{v \in V; f(u, v, -) \equiv 0\}$.
- The codimension of the radical of a vector u is called the rank of u, denoted by $r(u)$.
- Let F be the two-element field $GF(2)$ (any finite field) and let n denote the dimension of V, $n = \dim V$.
- The radical polynomial of a form f is defined by:

$$p(f) = \sum_{u \in V} x^{r(u)} y^{n-r(u)},$$

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Theorem

Let f be a bilinear alternating form on V of dimension n. Then it is equivalent to $g = b_1 \wedge b_2 + \ldots + b_{k-1} \wedge b_k, k \leq n$.

Notation: Let $B = \{b_1, \ldots, b_n\}$ be a basis of V. Alternating form $b_1 \wedge b_2$ is a form given by

$$(b_1 \wedge b_2)(b_1, b_2) = 1 \;\Rightarrow\; (b_1 \wedge b_2)(b_2, b_1) = -1,$$

$$(b_1 \wedge b_2)(b_i, b_j) = 0 \text{ whenever } \{i, j\} \neq \{1, 2\}.$$

Example of a trilinear form:

$$f = b_1 \wedge b_2 \wedge b_3 + b_3 \wedge b_4 \wedge b_5 = 123 + 345.$$

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Radical polynomials on dimension 7 :-(

Classification by A. M. Cohen and A. G. Helminck (1988):

0	$128y^7$
123	$16y^4(y^3+7x^2y)$
123 + 345	$4y^2(y^5+15x^2y^3+16x^4y)$
123 + 456	$2y(y^3+7x^2y)(y^3+7x^2y)$
123 + 345 + 156	$2y(y^6+7x^2y^4+56x^4y^2)$
123 + 234 + 345 + 246 + 156	$2y(y^6+63x^4y^2)$
123 + 145 + 167 + 357	$y^7+7x^2y^5+56x^4y^3+64x^6y$
123 + 167 + 246 + 357	$y^7+3x^2y^5+76x^4y^3+48x^6y$
123 + 145 + 167	$y^7+63x^2y^5+64x^6y$
123 + 345 + 567	$y^7+13x^2y^5+82x^4y^3+32x^6y$
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Orthogonal decompositions

Let V be a direct sum of its subspaces V_i ($V = \bigoplus V_i$), $1 \leq i \leq m$ and let f be a nondegenerate form. Let $\pi_i : V \to V_i$ be the projections with respect to this direct sum. Then the system of subspaces $\bigoplus V_i$ is an orthogonal decomposition of f if

$$f(u_1, \ldots, u_k) = \sum_i f(\pi_i(u_1), \ldots, \pi_i(u_k)).$$

A form f is indecomposable if it has no orthogonal decomposition with $m \geq 2$.

Example of a decomposable form:

$$f = b_1 \wedge b_2 \wedge b_3 + b_4 \wedge b_5 \wedge b_6 = 123 + 456.$$

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For an orthogonal decomposition $V = \bigoplus V_i$ of f denote by f_i the restriction of f to V_i. Then we shall write $f = \oplus f_i$.

Theorem

Let f be a decomposable form $f = \oplus f_i$. Then

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A Steiner triple system on n points is a system of three-element sets of points such that any pair of points is contained in exactly one set.

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Let S be a STS on n points. Define a trilinear alternating form f_S on n-dimensional vector space V (over a field F) with a basis $B = \{b_1, \ldots, b_n\}$:

$$f_S = \sum_{\{i,j,k\} \in S} b_i \wedge b_j \wedge b_k.$$

Call this form a Steiner form.

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Bad news

1.

n	Number of STS on n points
7	1 (Fano Plane)
9	1 (Affine Plane)
13	2
15	80
19	11084874829

2. $\mathrm{Aut}(\text{Fano Plane}) \neq \mathrm{Aut}(\mathrm{f}_{\text{Fano Plane}})$,
   $\mathrm{Aut}(f) = \{\phi \in \mathbf{GL}(V); f(u,v,w) = f(\phi(u),\phi(v),\phi(w))\}$.
3. $123+145+167+357+346+247+256 \sim 123+145+167+357$

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Radical polynomial of Steiner forms on dimension 13

$$1x^0 + 0x^2 + 0x^4 + 25x^6 + 476x^8 + 4634x^{10} + 3056x^{12}$$

$$1x^0 + 0x^2 + 0x^4 + 26x^6 + 442x^8 + 4615x^{10} + 3108x^{12}$$

Radical polynomial of Steiner forms on dimension 15 :-(

$$1x^0+15x^2+560x^4+448x^6+15360x^8+0x^{10}+0x^{12}+16384x^{14}$$
$$1x^0+7x^2+96x^4+568x^6+5472x^8+10240x^{10}+0x^{12}+16384x^{14}$$
$$1x^0+3x^2+40x^4+420x^6+3120x^8+8704x^{10}+8192x^{12}+12288x^{14}$$
$$1x^0+3x^2+20x^4+192x^6+2216x^8+8320x^{10}+8704x^{12}+13312x^{14}$$
$$1x^0+3x^2+28x^4+392x^6+2520x^8+8832x^{10}+8704x^{12}+12288x^{14}$$
$$1x^0+3x^2+4x^4+140x^6+1324x^8+7104x^{10}+12928x^{12}+11264x^{14}$$
$$1x^0+3x^2+4x^4+480x^6+2008x^8+6976x^{10}+15104x^{12}+8192x^{14}$$
$$1x^0+1x^2+14x^4+144x^6+1592x^8+6248x^{10}+13248x^{12}+11520x^{14}$$
$$1x^0+1x^2+8x^4+56x^6+950x^8+5320x^{10}+14912x^{12}+11520x^{14}$$
$$1x^0+1x^2+8x^4+90x^6+1012x^8+5736x^{10}+14912x^{12}+11008x^{14}$$
$$1x^0+1x^2+2x^4+30x^6+446x^8+4196x^{10}+17900x^{12}+10192x^{14}$$
$$1x^0+1x^2+6x^4+48x^6+719x^8+5313x^{10}+15672x^{12}+11008x^{14}$$
$$1x^0+1x^2+8x^4+126x^6+1344x^8+5304x^{10}+14208x^{12}+11776x^{14}$$
$$1x^0+1x^2+10x^4+120x^6+1452x^8+5456x^{10}+13696x^{12}+12032x^{14}$$

Radical polynomial of Steiner forms on dimension 15 :-|

$$1x^0 + 1x^2 + 4x^4 + 68x^6 + 834x^8 + 4916x^{10} + 16576x^{12} + 10368x^{14}$$
$$1x^0 + 1x^2 + 28x^4 + 266x^6 + 2312x^8 + 7504x^{10} + 9856x^{12} + 12800x^{14}$$
$$1x^0 + 1x^2 + 4x^4 + 150x^6 + 1292x^8 + 5112x^{10} + 16480x^{12} + 9728x^{14}$$
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$$1x^0 + 1x^2 + 0x^4 + 0x^6 + 205x^8 + 2883x^{10} + 19150x^{12} + 10528x^{14}$$
$$1x^0 + 0x^2 + 4x^4 + 21x^6 + 332x^8 + 3367x^{10} + 18103x^{12} + 10940x^{14}$$
$$1x^0 + 0x^2 + 4x^4 + 13x^6 + 292x^8 + 3110x^{10} + 17996x^{12} + 11352x^{14}$$
$$1x^0 + 0x^2 + 4x^4 + 25x^6 + 445x^8 + 3333x^{10} + 17856x^{12} + 11104x^{14}$$
$$1x^0 + 0x^2 + 5x^4 + 32x^6 + 513x^8 + 3760x^{10} + 17389x^{12} + 11068x^{14}$$
$$1x^0 + 0x^2 + 2x^4 + 15x^6 + 241x^8 + 2748x^{10} + 18841x^{12} + 10920x^{14}$$
$$1x^0 + 0x^2 + 2x^4 + 13x^6 + 222x^8 + 2668x^{10} + 18578x^{12} + 11284x^{14}$$

Radical polynomial of Steiner forms on dimension 15 :-()

$$1x^0 + 0x^2 + 4x^4 + 18x^6 + 381x^8 + 3145x^{10} + 17823x^{12} + 11396x^{14}$$
$$1x^0 + 0x^2 + 2x^4 + 7x^6 + 200x^8 + 2370x^{10} + 18884x^{12} + 11304x^{14}$$
$$1x^0 + 0x^2 + 4x^4 + 24x^6 + 376x^8 + 3423x^{10} + 17988x^{12} + 10952x^{14}$$
$$1x^0 + 0x^2 + 2x^4 + 6x^6 + 173x^8 + 2338x^{10} + 18716x^{12} + 11532x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 8x^6 + 133x^8 + 2090x^{10} + 18699x^{12} + 11836x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 9x^6 + 150x^8 + 2182x^{10} + 18973x^{12} + 11452x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 6x^6 + 146x^8 + 2000x^{10} + 19158x^{12} + 11456x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 6x^6 + 144x^8 + 1845x^{10} + 18748x^{12} + 12024x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 0x^6 + 102x^8 + 1981x^{10} + 19012x^{12} + 11672x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 5x^6 + 105x^8 + 1561x^{10} + 19196x^{12} + 11900x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 2x^6 + 138x^8 + 1758x^{10} + 18916x^{12} + 11952x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 4x^6 + 163x^8 + 1989x^{10} + 18774x^{12} + 11836x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 5x^6 + 162x^8 + 1991x^{10} + 18924x^{12} + 11684x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 105x^8 + 1660x^{10} + 18901x^{12} + 12100x^{14}$$

Radical polynomial of Steiner forms on dimension 15 :-)

$$1x^0 + 0x^2 + 0x^4 + 4x^6 + 147x^8 + 1721x^{10} + 18915x^{12} + 11980x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 4x^6 + 101x^8 + 1628x^{10} + 18778x^{12} + 12256x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 4x^6 + 102x^8 + 1630x^{10} + 19055x^{12} + 11976x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 87x^8 + 1534x^{10} + 19040x^{12} + 12104x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 0x^6 + 113x^8 + 1835x^{10} + 18806x^{12} + 12012x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 94x^8 + 1582x^{10} + 18765x^{12} + 12324x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 88x^8 + 1394x^{10} + 18740x^{12} + 12544x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 4x^6 + 95x^8 + 1707x^{10} + 18985x^{12} + 11976x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 3x^6 + 111x^8 + 1568x^{10} + 18925x^{12} + 12160x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 3x^6 + 107x^8 + 1559x^{10} + 18942x^{12} + 12156x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 2x^6 + 125x^8 + 1817x^{10} + 19090x^{12} + 11732x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 1x^6 + 131x^8 + 1828x^{10} + 18966x^{12} + 11840x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 3x^6 + 107x^8 + 1605x^{10} + 18852x^{12} + 12200x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 110x^8 + 1547x^{10} + 18968x^{12} + 12140x^{14}$$

Radical polynomial of Steiner forms on dimension 15 :-)

$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 83x^8 + 1500x^{10} + 18847x^{12} + 12336x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 1x^6 + 109x^8 + 1755x^{10} + 18869x^{12} + 12032x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 3x^6 + 147x^8 + 2000x^{10} + 18888x^{12} + 11728x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 108x^8 + 1707x^{10} + 19134x^{12} + 11816x^{14}$$
$$1x^0 + 1x^2 + 0x^4 + 0x^6 + 142x^8 + 2604x^{10} + 19796x^{12} + 10224x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 0x^6 + 106x^8 + 1721x^{10} + 18935x^{12} + 12004x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 7x^6 + 124x^8 + 1938x^{10} + 19145x^{12} + 11552x^{14}$$
$$1x^0 + 0x^2 + 1x^4 + 3x^6 + 107x^8 + 1794x^{10} + 19078x^{12} + 11784x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 91x^8 + 1482x^{10} + 18993x^{12} + 12200x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 81x^8 + 1575x^{10} + 19101x^{12} + 12008x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 78x^8 + 1624x^{10} + 19059x^{12} + 12004x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 89x^8 + 1459x^{10} + 19010x^{12} + 12208x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 79x^8 + 1506x^{10} + 19012x^{12} + 12168x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 5x^6 + 103x^8 + 1624x^{10} + 18867x^{12} + 12168x^{14}$$

Radical polynomial of Steiner forms on dimension 15 ;-)

$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 78x^8 + 1478x^{10} + 19006x^{12} + 12204x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 83x^8 + 1538x^{10} + 18881x^{12} + 12264x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 82x^8 + 1575x^{10} + 19276x^{12} + 11832x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 4x^6 + 101x^8 + 1790x^{10} + 18808x^{12} + 12064x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 3x^6 + 86x^8 + 1571x^{10} + 18847x^{12} + 12260x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 10x^6 + 120x^8 + 1907x^{10} + 18630x^{12} + 12100x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 1x^6 + 67x^8 + 1513x^{10} + 18978x^{12} + 12208x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 2x^6 + 74x^8 + 1465x^{10} + 18650x^{12} + 12576x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 6x^6 + 80x^8 + 1783x^{10} + 18722x^{12} + 12176x^{14}$$
$$1x^0 + 0x^2 + 0x^4 + 0x^6 + 45x^8 + 870x^{10} + 19100x^{12} + 12752x^{14}$$

Theorem (M. Computer, 2008)

Let n be at most 15. Then two STS on n points are isomorphic iff their Steiner forms over $\mathrm{GF}(2)$ are equivalent.

Thank you for your patience.