Markov bases of graph models of K_4-minor free graphs

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Graphs

Graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$

Cut - for a partition (A,B) of V define
$Cut(A, B) = \{\{i, j\}, i \in A, i \in B$ or $i \in B, j \in A\}$

Cut ideal

Homomorphism of polynomial rings $\pi_G : \mathbb{K}[q] \to \mathbb{K}[s,t]$

$$q_{A|B} \to \prod_{\{i,j\} \in Cut(A,B)} s_{ij} \prod_{\{i,j\} \notin Cut(A,B)} t_{ij}$$

Cut ideal of G $I_G = Ker \pi_G$.

Examples

$G = K_4$

$$q_{|1234} \longrightarrow t_{12}t_{13}t_{14}t_{23}t_{24}t_{34}$$
$$q_{12|34} \longrightarrow t_{12}s_{13}s_{14}s_{23}s_{24}t_{34}$$
$$q_{13|24} \longrightarrow s_{12}t_{13}s_{14}s_{23}t_{24}s_{34}$$
$$q_{14|23} \longrightarrow s_{12}s_{13}t_{14}t_{23}s_{24}s_{34}$$

$$q_{1|234} \longrightarrow s_{12}s_{13}s_{14}t_{23}t_{24}t_{34}$$
$$q_{2|134} \longrightarrow s_{12}t_{13}t_{14}s_{23}s_{24}t_{34}$$
$$q_{3|124} \longrightarrow t_{12}s_{13}t_{14}s_{23}t_{24}s_{34}$$
$$q_{4|123} \longrightarrow t_{12}t_{13}s_{14}t_{23}s_{24}s_{34}$$

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$$I_{K_4} = \langle q_{|1234}q_{12|34}q_{13|24}q_{14|23} - q_{1|234}q_{2|134}q_{3|124}q_{4|123} \rangle$$

Examples

$G = C_4$, $E = \{12, 23, 34, 14\}$

π_{C_4} derived from π_{K_4} by setting $s_{13} = t_{13} = s_{24} = t_{24} = 1$

$$I_{C_4} = \langle \begin{array}{l} q_{|1234}q_{13|24} - q_{1|234}q_{3|124}, \\ q_{|1234}q_{13|24} - q_{2|134}q_{4|123}, \\ q_{|1234}q_{13|24} - q_{12|34}q_{14|23} \end{array} \rangle$$

Binary graph models

Homomorphism of polynomial rings

$\phi_G : \mathbb{K}[p_{i_1,\ldots,i_n} | i_j \in \{0,1\}] \longrightarrow \mathbb{K}[q^{(k\ell)}_{i_k i_\ell} | (k\ell) \in E, i_k, i_\ell \in \{0,1\}]$

$$p_{i_1,\ldots,i_n} \longrightarrow \prod_{(k\ell)\in E} q^{(k\ell)}_{i_k i_\ell}$$

Ideal of a binary graph model $J_G = Ker(\phi_G)$.

Examples

$G = K_3$

$$\begin{array}{ll} p_{000} \longrightarrow q^{12}_{00}\ q^{13}_{00}\ q^{23}_{00} & p_{001} \longrightarrow q^{12}_{00}\ q^{13}_{01}\ q^{23}_{01} \\ p_{011} \longrightarrow q^{12}_{01}\ q^{13}_{01}\ q^{23}_{11} & p_{010} \longrightarrow q^{12}_{01}\ q^{13}_{00}\ q^{23}_{10} \\ p_{101} \longrightarrow q^{12}_{10}\ q^{13}_{11}\ q^{23}_{01} & p_{100} \longrightarrow q^{12}_{10}\ q^{13}_{10}\ q^{23}_{00} \\ p_{110} \longrightarrow q^{12}_{11}\ q^{13}_{10}\ q^{23}_{10} & p_{111} \longrightarrow q^{12}_{11}\ q^{13}_{11}\ q^{23}_{11} \end{array}$$

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$$J_{K_3} = \langle p_{000}p_{011}p_{101}p_{110} - p_{001}p_{010}p_{100}p_{111} \rangle$$

Markov basis

Let $\psi_{ij} : \mathbb{Z}^{2^n} \to \mathbb{Z}^{2^2}$ be given by $e_{a_1,\ldots,a_n} \to e_{a_i a_j}$ and let ψ_G be the product of ψ_{ij}'s over all $\{ij\} \in E$.

Markov basis $B \subseteq Ker(\psi_G)$:

$$\forall v_1, v_2 \in \mathbb{N}^{2^n}, \psi_G(v_1) = \psi_G(v_2) : \; \exists u_1, \ldots, u_\ell \in \pm B$$

$$v_1 + \sum_{k=1}^{\ell} u_k = v_2 \text{ and } v_1 + \sum_{k=1}^{\ell'} u_k \in \mathbb{N}^{2^n} \text{ for } 1 \leq \ell' \leq \ell$$

Markov basis

Theorem [Diaconis and Sturmfels]
A finite subset $B = \{u_1, \ldots, u_k\} \in \pm Ker(\psi_G)$ is a Markov basis iff the set of binomials $\{p^{u_i^+} - p^{u_i^-}\}$ is a generating set of J_G.

Markov width

$\mu(\star) =$ largest degree of a minimal genarator of the ideal

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$$I_{C_4} =< q_{|1234}q_{13|24} - q_{1|234}q_{3|124}, \ldots >, \quad \mu(I_{C_4}) = 2$$

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$$\begin{aligned} I_{K_4} &= < q_{|1234}q_{12|34}q_{13|24}q_{14|23} - q_{1|234}q_{2|134}q_{3|124}q_{4|123} > \\ J_{K_3} &= < p_{000}p_{011}p_{101}p_{110} - p_{001}p_{010}p_{100}p_{111} > \end{aligned}$$

$$\mu(I_{K_4}) = \mu(J_{K_3}) = 4$$

Combinatorial interpretation

$q_{A\|B}$	$Cut(A,B)$
$q_{A_1\|B_1} q_{A_2\|B_2} \cdots$	collection of cuts
	$\prod_{(k\ell)\in E} s_{k\ ell}^{a} t_{k\ ell}^{b}$
$p_{i_1,\ldots,i_n}$	$c : V \longrightarrow \{0,1\}$
$p_{i_1,\ldots,i_n} p_{j_1,\ldots,j_n} \cdots$	collection of 2-colorings
	$\prod_{(k\ell)\in E} \left(q_{00}^{(k\ell)}\right)^{a} \left(q_{01}^{(k\ell)}\right)^{b} \left(q_{10}^{(k\ell)}\right)^{c} \left(q_{11}^{(k\ell)}\right)^{d}$

Universal vertex

2-coloring $\longrightarrow$ cut $\longrightarrow$ two possible 2-colorings

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$$\mu(J_G) = \mu(I_{\widehat{G}})$$

Bounds

G is K_3 – free	$\Leftrightarrow$	$\mu(J_G) \leq 2$	
G is K_4 – free	$\Leftrightarrow$	$\mu(I_G) \leq 2$	Engrsröm
	$\Leftrightarrow$	$\mu(J_G) \leq 4$	OUR RESULT
G is K_5 – free	$\Leftrightarrow$	$\mu(I_G) \leq 4$	conjectured by S&S

Other conjectures

G planar graph $\Rightarrow$ $\mu(J_G) \leq 6$

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$$\mu(J_G) = f(tw(G))$$