On some properties of quasiorder lattices of monounary algebras

Mária Petrejčíková   Danica Jakubíková-Studenovská

P.J.Šafárik University, Košice, Slovakia

SSAOS, Třešť, 2008

Basic concepts and remarks

$\mathcal{A} = (A, f)$
monounary algebra

- for $x, y \in A$ we put $x \sim y$ if there are $n, m \in N \cup \{0\}$ such that $f^n(x) = f^m(y)$
- elements of $A/\sim$ are called **connected components** of (A, f)
- (A, f) is **connected** if it has only one connected component
- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in N$
- the set of all cyclic elements of some connected component of (A, f) is a **cycle** of (A, f)

Basic concepts and remarks

$\mathcal{A} = (A, f)$

monounary algebra

α

quasiorder of (A, f)

reflexive, transitive and compatible with all operations of (A, f)

$\mathcal{Quord}\ (A, f)$

all quasiorders of (A, f)

$(\mathcal{Quord}\ (A, f), \subseteq)$

lattice of all quasiorders of (A, f)

$I = \{(a, a) : a \in A\}$ ... the smallest quasiorder

A^2 ... the greatest quasiorder

Primitive lattices

J. Ježek and V. Slavík: Primitive lattices, Czech.M.J., 1979.

Definition

A lattice L is called **primitive**, *if the class of all lattices that do* **not contain** *a sublattice isomorphic to L is a variety.*

Example: pentagon ... variety of modular lattices
primitive lattice

J. Ježek and V. Slavík: Some examples of primitive lattices, Mathematica et physica, 1973.

Theorem

A lattice L is primitive iff it is non-trivial (i.e. of cardinality ≥ 2), finite, subdirectly irreducible and satisfies the following condition:
Whenever there exists a homomorphism of some lattice A onto L, then A contains a sublattice isomorphic to L.

Primitive lattices

11.2. Theorem. *The following lattices are (up to isomorphism) just the only primitive lattices:*

(i) A_1.
(ii) A_5.
(iii) $Z(R(A_5); c_{A_5}; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(iv) $Z(I_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(v) $Z(I_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(vi) $Z(J_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(vii) $Z(J_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(viii) A_2.
(ix) $Z(R(A_2); c_{A_2}; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(x) $Z(H_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(xi) $Z(H_n^*; b_n, e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(xii) $Z(K_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(xiii) $Z(K_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.
(xiv) $Z(A_4; 3; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 3$ *if* $k \neq 0$.

I_n

(xv) $Z(A_4; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 3$.

(xvi) $Z(B_n; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 3$ *if* $k \neq 0$.

(xvii) $Z(B_n; 10; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 3$.

(xviii) $Z(B_n^*; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 2$ *if* $k \neq 0$.

(xix) $Z(B_n^*; 10; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 2$

(xx) $Z(C_n; d_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 3$ *if* $k \neq 0$.

(xxi) $Z(C_n; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 3$.

(xxii) $Z(C_n^*; d_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 2$ *if* $k \neq 0$.

(xxiii) $Z(C_n^*; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 2$.

(xxiv) $Z(D_n; 6; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxv) $Z(D_0; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 2$.

(xxvi) $Z(D_n^*; 6; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxvii) $Z(D_0^*; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 3$.

(xxviii) $Z(E_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxix) $Z(E_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxx) $Z(F_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxxi) $Z(F_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxxii) $Z(G_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$. $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

(xxxiii) $Z(G_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 = 1$ *if* $k \neq 0$.

Small algebras and primitive sublattices

small algebras...(A, f), $|A| = 1$ or 2 or 3

How look like $(\mathcal{Quord}\,(A, f), \subseteq)$?

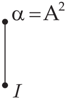
Which types of primitive lattices are contained by $(\mathcal{Quord}\,(A, f), \subseteq)$?

Small algebras and primitive sublattices

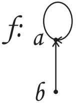
$\mathcal{A} = (A, f)$ ... monounary algebra

- if $|A| = 1$, then $\mathcal{Q}uord\ (A, f)$ is a 1-element lattice
- if $|A| = 2$ and
  - (A, f) is a 2-element cycle, then $\mathcal{Q}uord\ (A, f)$ is a 2-element lattice

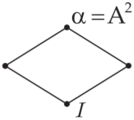
$(\mathcal{Q}uord(A, f), \subseteq)$:



  - (A, f) is not a cycle, then $\mathcal{Q}uord\ (A, f) \cong M_2$



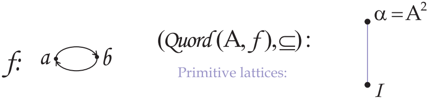
$(\mathcal{Q}uord(A, f), \subseteq)$:



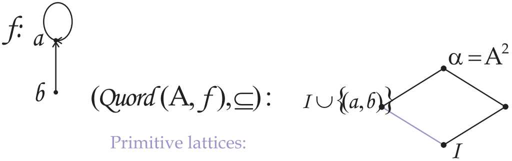
Small algebras and primitive sublattices

$\mathcal{A} = (A, f)$ ... monounary algebra

- if $|A| = 1$, then $\mathcal{Quord}\,(A, f)$ is a 1-element lattice
- if $|A| = 2$ and
    - (A, f) is a 2-element cycle, then $\mathcal{Quord}\,(A, f)$ is a 2-element lattice

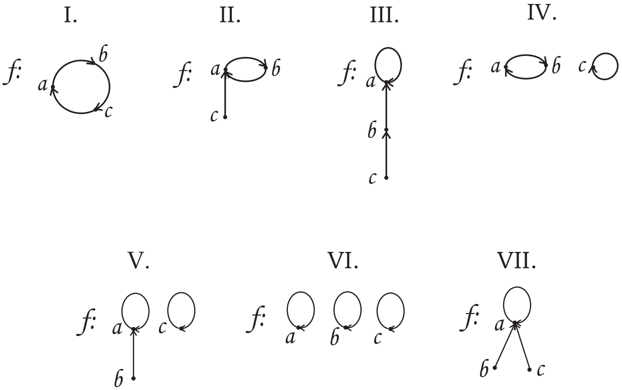


- (A, f) is not a cycle, then $\mathcal{Quord}\,(A, f) \cong M_2$

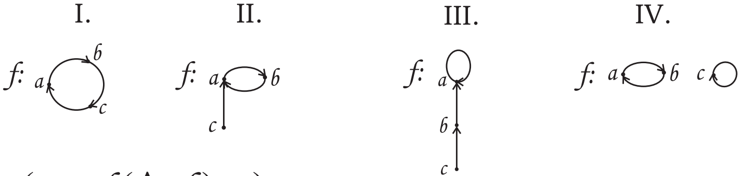


Small algebras and primitive sublattices

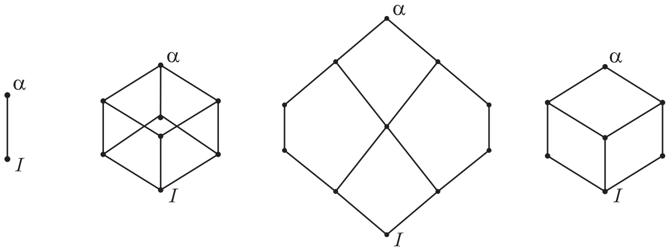
- if $|A| = 3$ and an unary operation f is:



Small algebras and primitive sublattices

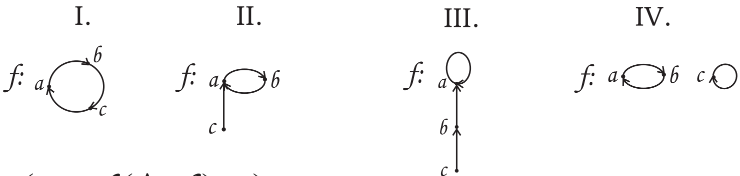


$(Quord(A, f), \subseteq)$:



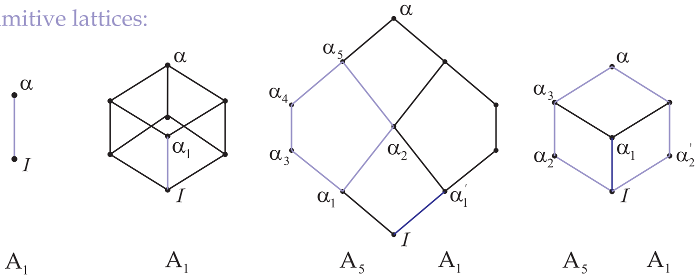
(11 quasiorders)

Small algebras and primitive sublattices



$(Quord(A, f), \subseteq)$:

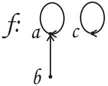
Primitive lattices:



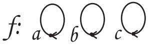
(11 quasiorders)

Small algebras and primitive sublattices

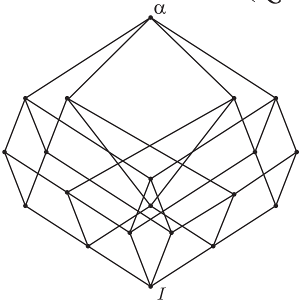
V.

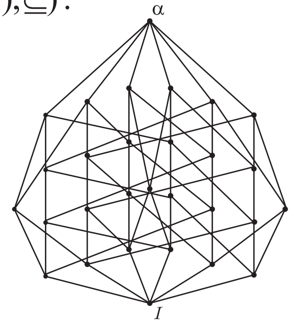


VI.



$(Quord(A, f), \subseteq)$:



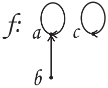


(20 quasiorders)

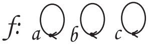
(29 quasiorders)

Small algebras and primitive sublattices

V.

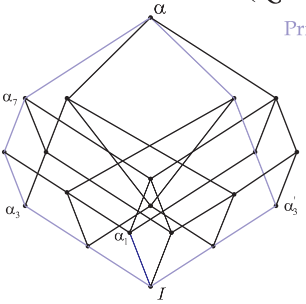


VI.



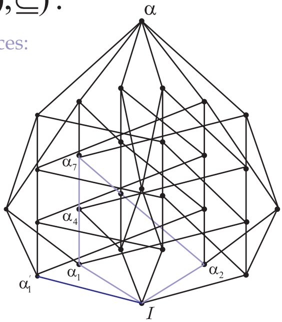
$(Quord(A, f), \subseteq)$:

Primitive lattices:



A_1 A_5

(20 quasiorders)



A_1 A_5

(29 quasiorders)

Summary

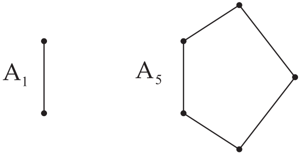
Let $\mathcal{A} = (A, f)$ be an monounary algebra of three elements.

Theorem

If $\mathcal{A}$ doesn't contain any singleton (an 1-element cycle), then $(\mathcal{Quord}\,(A, f), \subseteq)$ contains only one type of primitive sublattice, namely A_1.

Theorem

If $\mathcal{A}$ contains an 1-element cycle, $(\mathcal{Quord}\,(A, f), \subseteq)$ contains just two types of primitive sublattices, namely A_1 and A_5.



Complementarity of $\mathcal{Quord}(A, f)$

Assumption:

$\mathcal{A} = (A, f)$ ... monounary algebra

$\mathcal{Quord}\,(A, f)$ ... complementary lattice

Lemma 1

If $\mathcal{B} = (B, f)$ is a subalgebra of $\mathcal{A}$, then $\mathcal{Quord}\,(B, f)$ also is a complementary lattice.

Proof:

$$\beta \in \mathcal{Quord}\,(B, f) \Rightarrow \beta \cup I_A = \alpha \in \mathcal{Quord}\,(A, f) \Rightarrow$$
$$\exists\ \alpha' \in \mathcal{Quord}\,(A, f) \Rightarrow$$
$$\Rightarrow \text{denote } \beta' = \alpha' \cap B^2 \Rightarrow$$
$$\Rightarrow \beta' \in \mathcal{Quord}\,(B, f) \text{ and } \beta \vee \beta' = B^2,\ \beta \wedge \beta' = I_B,$$
i.e. β' is a complement of a β in $\mathcal{Quord}\,(B, f)$.$\Diamond$

I_A ... the smallest quasiorder of $\mathcal{Quord}\,(A, f)$,

$\alpha \vee \alpha' = A^2$, $\alpha \wedge \alpha' = I_A$

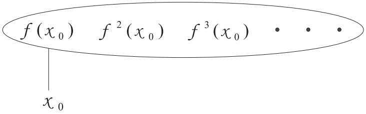
Complementarity of $\mathcal{Q}uord\,(A, f)$

Lemma 2

If $x_0 \in A$, then exists $m \in N$ such that $f^{m+1}(x_0) = f(x_0)$.

Proof:

Let $x_0 \in A$, arbitrary. We take $\alpha \in \mathcal{Q}uord\,(A, f)$ such that:



$\mathcal{Q}uord\,(A, f)$ ...complementary lattice $\Rightarrow \exists \alpha' \in \mathcal{Q}uord\,(A, f)$, such that $\alpha \vee \alpha' = A^2$ and $\alpha \wedge \alpha' = I_A$.

How look like complementary quasiorder α'?

$\forall a \in A\ \exists i$: $f^i(x_0)\alpha' a$, because $\alpha \vee \alpha' = A^2$

$\downarrow$

also for x_0, $x_0 \in A\ \exists i$, such that:

$f^i(x_0)\alpha' x_0 \Rightarrow f^{i+1}(x_0)\alpha' f(x_0) \Rightarrow f^{i+1}(x_0)\ \alpha \wedge \alpha' f(x_0)$.

$\alpha \wedge \alpha' = I_A$ and so $f^{i+1}(x_0) = f(x_0)$.$\diamondsuit$

Mária Petrejčíková, Danica Jakubíková-Studenovská

Complementarity of $\mathit{Quord}\,(A, f)$

Lemma 3

All cycles have the same length.

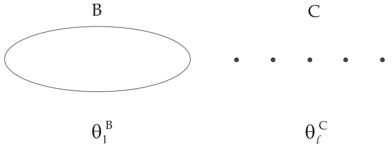
Proof:

Suppose, on the contrary $k \neq \ell$; $k, \ell \in N$ and

k	number of elements of cycle	ℓ
B	cycles of $\mathcal{A}$	C

Let $k < \ell$,

- ℓ isn't a multiple of k. We take a binary relation α:



$\alpha = I_A \cup \theta_1^B \cup \theta_\ell^C$ ref., tran., compat.$\Rightarrow$ $\alpha \in \mathit{Quord}\,(A, f)$

Complementarity of $\mathcal{Q}uord\,(A, f)$

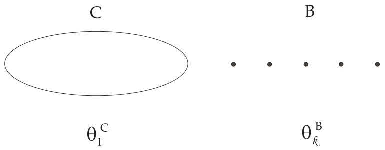
$\mathcal{Q}uord\,(A, f)$ . . . complementary lattice $\Rightarrow \exists\alpha' \in \mathcal{Q}uord\,(A, f)$, α' complement of α.

How look like complementary quasiorder α'?

$\vdots$

$\nexists\alpha'$ complement of α
(in contradiction with complementarity of $\mathcal{Q}uord\,(A, f)$)

- ℓ is a multiple of k, i.e. $\exists u \in N - \{1\}$; $\ell = u \cdot k$. We take a binary relation α:



$\alpha = I_A \cup \theta_k^B \cup \theta_1^C$ ref., tran., compat.$\Rightarrow$ $\alpha \in \mathcal{Q}uord\,(A, f)$

How look like complementary quasiorder α'?

$\vdots$

Complementarity of $\mathcal{Q}uord\,(A, f)$

Lemma 4

If a subalgebra $\mathcal{B}$ is cycle of n elements, then n is square-free.

Complementarity of $\mathcal{Quord}\,(A, f)$

Theorem

Let $\mathcal{A} = (A, f)$ be a monounary algebra. If $\mathcal{Quord}\,(A, f)$ is a complementary lattice, then the following conditions are satisfied:

1. *Each connected component of (A, f) contains a cycle.*
2. *There is $n \in N$ such that each cycle of (A, f) has n elements.*
3. *The number n is square-free.*
4. *The element $f(a)$ is cyclic, for each $a \in A$.*

THANK YOU FOR YOUR ATTENTION!

Fig. 1: A_1

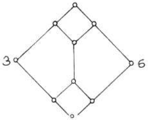


Fig. 2: A_2

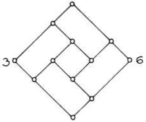


Fig. 4: A_4

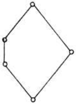


Fig. 18: A_5

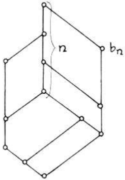


Fig. 19: J_n

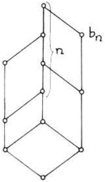


Fig. 20: K_n

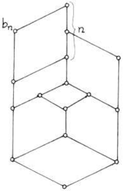


Fig. 16: H_n

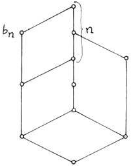


Fig. 17: I_n

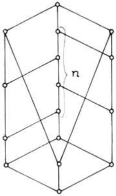


Fig. 10: G_n

△

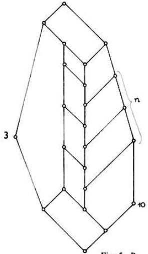


Fig. 5: B_n

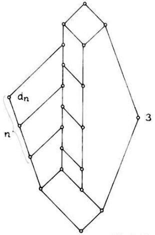


Fig. 6: C_n

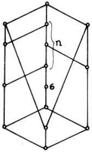


Fig. 7: D_n

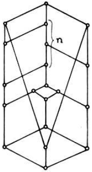


Fig. 8: E_n

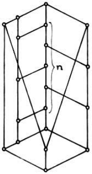


Fig. 9: F_n

Mária Petrejčíková, Danica Jakubíková-Studenovská