On literal varieties of languages

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Syntactic structures

Let $L \subseteq A^*$ be a regular language. We define the following relations on A^* and $A^\Box$ = all finite subsets of A^*, respectively :

for $u, v, u_1, \ldots, u_k, v_1, \ldots, v_\ell \in A^*$,

$$u \sim_L v \text{ if and only if } (\, \forall\, x, y \in A^* \,)(\, xuy \in L \Leftrightarrow xvy \in L \,)\ ,$$

$\{u_1, \ldots, u_k\} \approx_L \{v_1, \ldots, v_\ell\}$ if and only if

$$(\, \forall\, x, y \in A^* \,)(\, xu_1y, \ldots, xu_ky \in L \Leftrightarrow xv_1y, \ldots, xv_\ell y \in L \,)\ .$$

The quotient structures

$$(\mathrm{O}(L), \cdot) = (A^*, \cdot)/\sim_L \text{ and } (\mathrm{S}(L), \cdot, \vee) = (A^\Box, \cdot, \cup)/\approx_L$$

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The assignments

$$\phi_L : u \mapsto u \sim_L \text{ and } \psi_L : \{u_1, \ldots, u_k\} \mapsto \{u_1, \ldots, u_k\} \approx_L$$

are called syntactic monoid/semiring homomorphisms.

The monoid $(O(L), \cdot)$ is ordered by the relation

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Eilenberg type theorems

Theorem

(i) (Eilenberg) Boolean varieties of languages correspond to pseudovarieties of finite monoids. Here $L \mapsto$ syntactic monoid of L.

(ii) (Ésik & Co., Straubing) Literal boolean varieties of languages correspond to literal pseudovarieties of homomorphisms from finitely generated free monoids onto finite monoids. Here $L \mapsto$ syntactic homomorphism of L.

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“Varieties” of languages

A class of (regular) languages is an operator $\mathcal{V}$ assigning to each finite set A a set $\mathcal{V}(A)$ of regular languages over the alphabet A. Such a class is a positive variety if

(0) for each A, we have $\emptyset, A^* \in \mathcal{V}(A)$,

(i) each $\mathcal{V}(A)$ is closed with respect to finite unions, finite intersections and quotients, and

(ii) for each finite sets A and B and a homomorphism $f : B^* \to A^*$, $K \in \mathcal{V}(A)$ implies $f^{-1}(K) \in \mathcal{V}(B)$.

Adding the condition

(iii) each $\mathcal{V}(A)$ is closed with respect to complements,

we get a boolean variety.
A modification of (ii) to

(ii') for each finite sets A and B and a homomorphism $f : B^* \to A^*$ with $f(B) \subseteq A$, $K \in \mathcal{V}(A)$ implies $f^{-1}(K) \in \mathcal{V}(B)$

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C-universal algebra

Let $\boldsymbol{V}$ be a variety of algebras of a fixed signature. Let W_A be the free $\boldsymbol{V}$-algebra over the set A. We consider a category C of free $\boldsymbol{V}$-algebras, that is, the objects are all W_A's for sets A, and the homsets $\mathrm{C}(W_B, W_A)$ consist of certain homomorphisms from W_B into W_A.

Basic example :

$\boldsymbol{V}$ = all monoids, $W_A = A^*$, $p \in \mathrm{C}_{\mathrm{lit}}(B^*, A^*)$ iff, for each $b \in B$, $p(b) \in A$ - literal homomorphisms.

Let

$$\mathfrak{V} = \{\ \phi : W_A \twoheadrightarrow S \mid A \text{ is a set and } S \in \boldsymbol{V}\ \}$$

be the class of all surjective homomorphisms from free $\boldsymbol{V}$-algebras onto $\boldsymbol{V}$-algebras; we speak about $\boldsymbol{V}$-homomorphisms.

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Let $\mathfrak{U} \subseteq \mathfrak{V}$. We define :

$$\mathrm{H}\mathfrak{U} = \{\, \sigma\phi : W_A \twoheadrightarrow T \mid$$

$$T \in \boldsymbol{V},\ (\phi : W_A \twoheadrightarrow S) \in \mathfrak{U},\ \sigma : S \twoheadrightarrow T \text{ a surj. homom.} \,\},$$

$$\mathrm{S}_{\mathrm{C}}\mathfrak{U} = \{\, \phi p : W_B \twoheadrightarrow \mathrm{im}(\phi p) \mid$$

$$B \text{ a set},\ p \in \mathrm{C}(W_B, W_A),\ (\phi : W_A \twoheadrightarrow S) \in \mathfrak{U} \,\},$$

$$\mathrm{P}\,\mathfrak{U} = \{\, (\phi_\gamma)_{\gamma\in\Gamma} : W_A \twoheadrightarrow \mathrm{im}((\phi_\gamma)_{\gamma\in\Gamma}) \mid$$

$$A, \Gamma \text{ sets},\ (\phi_\gamma : W_A \twoheadrightarrow S_\gamma) \in \mathfrak{U} \text{ for } \gamma \in \Gamma \,\},$$

$$(\phi_\gamma)_{\gamma\in\Gamma} : W_A \to \prod_{\gamma\in\Gamma} S_\gamma,\ u \mapsto (\phi_\gamma(u))_{\gamma\in\Gamma}\ .$$

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We define the generalized C-varieties as classes of $\boldsymbol{V}$-homomorphisms $\mathfrak{U} \subseteq \mathfrak{V}$ closed with respect to H, S_{C}, P_{f} (products of finite families) and Po_{C}.
Similarly, C-pseudovarieties of finite $\boldsymbol{V}$-homomorphisms are classes $\mathfrak{X} \subseteq \mathrm{Fin}\mathfrak{V}$ closed with respect to H, S_{C}, and P_{f}.
An n-ary $\boldsymbol{V}$-identity is a pair $u = v$ where $u, v \in W_n$.
A $\boldsymbol{V}$-homomorphism $\phi : W_A \twoheadrightarrow S$ C-satisfies $u = v$ if

$$(\forall\, p \in \mathrm{C}(W_n, W_A)\,)\ (\phi p)(u) = (\phi p)(v)\,.$$

The invention of equational logic for C-pseudovarieties of finite $\boldsymbol{V}$-homomorphisms was possible only after a deep understanding of categories of such homomorphisms - see Kunc. Here we recall only the right definition of morphisms :

$$\sigma : (\phi : W_A \twoheadrightarrow S) \to (\psi : W_B \twoheadrightarrow T)$$

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Theorem

1. (LP) $\mathbb{C}$-varieties of $\boldsymbol{V}$*-homomorphisms are exactly classes of homomorphisms determined by* $\boldsymbol{V}$*-identities.*

2. (LP) Generalized $\mathbb{C}$*-varieties of* $\boldsymbol{V}$*-homomorphisms are exactly directed unions of* $\mathbb{C}$*-varieties of* $\boldsymbol{V}$*-homomorphisms.*

3. (LP) $\mathbb{C}$*-pseudovarieties of finite* $\boldsymbol{V}$*-homomorphisms are exactly classes of the form* $\mathrm{Fin}\,\mathfrak{U}$ *where* $\mathfrak{U}$ *is a directed union of* $\mathbb{C}$*-varieties of* $\boldsymbol{V}$*-homomorphisms. If the type is finite we can restrict ourselves to unions of chains.*

4. (Kunc) $\mathbb{C}$*-pseudovarieties of finite* $\boldsymbol{V}$*-homomorphisms are exactly classes of finite* $\boldsymbol{V}$*-homomorphisms* $\mathbb{C}$*-determined by* $(\mathrm{Fin}\,\boldsymbol{V})$*-pseudoidentities.*

L.P., On varieties, generalized varieties and pseudovarieties of homomorphisms, Contributions to General Algebra 16, Verlag Johannes Heyn, Klagenfurt 2005, pp. 173-187

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Varieties of group languages

Languages over $X_n = \{x_1, \dots, x_n\}$ corresponding to finite members of certain varieties of groups are well-known :

1. Boolean combinations of

$$\{u \in X_n^* \mid |u|_i \equiv \ell' \bmod \ell\},\ i \in \{1, \dots, n\},\ \ell \in \mathbb{N},\ \ell' \in \{0, \dots, \ell - 1\}$$

for the class of all abelian groups.

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4. Boolean combinations of

$$\{u \in X_n^* \mid \binom{u}{v} \equiv r' \bmod r\},\ v \in X_n^*,\ |v| \leq c,\ r \in \mathbb{N},\ r' \in \{0, \ldots, r-1\}$$

for the class of all nilpotent groups of class $\leq c$.

Such results can be refined as follows :

1'. Disjoint unions

$$\{u \in X_n^* \mid |u|_1 \equiv \ell_1, \ldots, |u|_n \equiv \ell_n \bmod \ell\},\ \ell \in \mathbb{N},\ \ell_1, \ldots, \ell_n \in \{0, \ldots, \ell - 1\}\ ,$$

ℓ fixed, for the class of all abelian groups.

It is not difficult to refine the results 2,3 and 4 in a similar way.

Our goal :

A. Find all literal varieties of homomorphisms onto abelian groups and describe the corresponding languages (in the finer form) – done 2005.

B. Find all (or at least some hierarchy) of literal varieties of homomorphisms onto nilpotent groups of class ≤ 2 and describe the corresponding languages.

4. Boolean combinations of

$$\{u \in X_n^* \mid \binom{u}{v} \equiv r' \bmod r\},\ v \in X_n^*,\ |v| \le c,\ r \in \mathbb{N},\ r' \in \{0, \ldots, r-1\}$$

for the class of all nilpotent groups of class $\le c$.

Such results can be refined as follows :

1'. Disjoint unions

$$\{u \in X_n^* \mid |u|_1 \equiv \ell_1, \ldots, |u|_n \equiv \ell_n \bmod \ell\},\ \ell \in \mathbb{N},\ \ell_1, \ldots, \ell_n \in \{0, \ldots, \ell-1\}\ ,$$

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Literal varieties of homomorphisms onto abelian groups and the corresponding languages

Our basic ingredients are the following languages :
Let $n, \ell, k \in \mathbb{N}$ with $k \mid \ell$,
let $\ell' \in \{0, \ldots, \ell - 1\}$,
let $k_1, \ldots, k_n \in \{0, \ldots, k - 1\}$ satisfy $k_1 + \cdots + k_n \equiv \ell' \bmod k$,

$$L(n; \ell, \ell'; k, k_1, \ldots, k_n) =$$

$$= \{\ u \in X_n^* \mid |u| \equiv \ell' \bmod \ell,\ |u|_1 \equiv k_1, \ldots, |u|_n \equiv k_n \bmod k\ \}\ .$$

Theorem (LP)

The following are pairwise different literal varieties of homomorphisms from free monoids onto abelian groups :

$$\mathscr{V}(\ell,k) = \mathrm{Mod_{lit}}(\ xy = yx,\ x^\ell = 1,\ x^k = y^k\ )$$

where $\ell, k \in \mathbb{N},\ k \mid \ell$. *The corresponding literally invariant congruences on* X^* *are of the form*

$$\rho(\ell,k) = \{\ (u,v) \in X^* \times X^* \mid |u| \equiv |v| \bmod \ell,$$

$$|u|_i \equiv |v|_i \bmod k \text{ for } i \in \mathbb{N}\}.$$

For fixed ℓ, k, *the corresponding languages on* X_n *are exactly the disjoint unions of* $L(n;\ell,\ell';k,k_1,\ldots,k_n)$.

L.P., Literal varieties and pseudovarieties of homomorphisms onto abelian groups, Proc. Int. Conf. on Semigroups and Languages, Lisboa 2005, World Scientific Publishing, Singapore 2007, pp. 255-264

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Literal varieties of homomorphisms onto nilpotent groups and the corresponding languages

Our basic ingredients are the following languages :
Let $n, \ell, k, r \in \mathbb{N}$ with $r \mid k \mid \ell$,
let $\ell' \in \{0, \ldots, \ell - 1\}$,
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let $r_{j,i} \in \{0, \ldots, r-1\}$ for $1 \leq i < j \leq n$. We put

$$L(n; \ell, \ell'; k, k_1, \ldots, k_n; r, r_{2,1}, \ldots, r_{n,1}, \ldots, r_{n,n-1}) =$$

$$= \{\ u \in X_n^* \mid |u| \equiv \ell' \text{ mod } \ell,\ |u|_1 \equiv k_1, \ldots, |u|_n \equiv k_n \text{ mod } k,$$
$$|u|_{j,i} \equiv r_{j,i} \text{ mod } r \text{ for all } 1 \leq i < j \leq n\ \}\ .$$

Theorem (OK & LP)

The following are pairwise different literal varieties of homomorphisms from free monoids onto nilpotent groups of class ≤ 2 :

$$\mathscr{V}(\ell,k,r) = \mathrm{Mod}_{\mathrm{lit}}(\ [x,[y,z]] = 1,\ x^{\ell} = 1,\ x^{k} = y^{k},\ [x,y]^{r} = 1\ )$$

where $\ell, k, r \in \mathbb{N},\ r \mid k \mid \ell$.

The corresponding literally invariant congruences on X^* are of the form

$$\rho(\ell,k,r) = \{\ (u,v) \in X^* \times X^* \mid |u| \equiv |v| \bmod \ell,$$

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For fixed ℓ, k, r, the corresponding languages on X_n are exactly the disjoint unions of

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Reminding type of identities :

$$([x,y][y,z][z,x])^{\alpha}=1,\ x^{\beta}y^{-\beta}=[x,y]^{\gamma}\ .$$

Literally idempotent languages and their varieties

A regular language L over a finite alphabet A is literally idempotent if its syntactic homomorphism $\phi_L : A^* \to \mathrm{O}(L)$ satisfies the pseudoidentity $x^2 = x$ literally, which means

$$(\forall\, a \in A)\ a^2 \sim_L a$$

or equivalently

$$(\forall\, u, v \in A^*,\ a \in A)\ (\, uav \in L \Leftrightarrow ua^2v \in L\,)\,. \qquad (*)$$

We denote the class of all such languages by $\mathscr{L}$.

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We can introduce a string rewriting system which is given by rules $pa^2q \to paq$ for each $a \in A$, $p, q \in A^*$. Let $\to^*$ be the reflexive-transitive closure of the relation $\to$. We say that a word $u \in A^*$ is the normal form of a word w if it satisfies the properties

$$w \to^* u \quad \text{and} \quad (\, u \to^* v \text{ implies } u = v\,)\,.$$

This system is confluent and terminating. Consequently, for any word $w \in A^*$, there exists the unique normal form $\overrightarrow{w} \in A^*$ of the word w. We will denote by $\sim$ the equivalence relation on A^* generated by the relation $\to$. In fact, it is a congruence on A^*.

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For any language $L \subseteq A^*$, we define

$$\overline{L} = \{ w \in A^* \mid (\exists\, u \in L)\; u \sim w \}$$

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Lemma

For a regular $L \subseteq A^*$, the language $\overline{L}$ is regular, too.

A complete deterministic automaton $\mathscr{A} = (Q, A, \cdot, i, T)$ is called literally idempotent if for each $q \in Q$ and $a \in A$ we have $q \cdot a^2 = q \cdot a$.

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Lemma

For a regular $L \subseteq A^*$, the following statements are equivalent :

(i) *L is literally idempotent,*

(ii) $\overline{L} = L$,

(iii) $\sim \subseteq \sim_L$,

(iv) *L is accepted by a literally idempotent complete deterministic finite automaton,*

(v) *the (canonical) minimal DFA for L is literally idempotent,*

(vi) *L is a (disjoint) union (not necessarily finite !) of the languages of the form*

$$a_1^+ a_2^+ \dots a_k^+,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,\ a_1 \neq a_2 \neq \dots \neq a_k\ .$$

The literally idempotent languages over $A = \{a\}$ are exactly: $\emptyset, \{1\}, a^+$ and a^*.

Now consider a regular language L over $A = \{a\}$ with the minimal deterministic automaton $\mathscr{A} = (Q, A, \cdot, i, T)$. Choose the minimal $d \in \mathbb{N}$ and then the minimal $k \in \mathbb{N}_0$ such that $i \cdot a^k = i \cdot a^{k+d}$. Let

$$M = L \cap \{1, a, \ldots, a^{k-1}\} \text{ and } N = L \cap a^k\{1, a, \ldots, a^{d-1}\} \ .$$

Then $M \cup N(a^d)^*$ is a "canonical" regular expression for L.

The situation for literally idempotent languages over $A = \{a, b\}$ is similar. Each regular language L over A is a disjoint union of the sets

$$L \cap (a\{a, b\}^* a \cup a),\ L \cap a\{a, b\}^* b,$$

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If L is literally idempotent each of the first four summands behaves similarly as a regular language over a single letter alphabet.

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If L is literally idempotent each of the first four summands behaves similarly as a regular language over a single letter alphabet.

We consider the first summand (the reasonings about the remaining ones are analogous). Let $\mathscr{A} = (Q, A, \cdot, i, T)$ be the minimal deterministic automaton for L. Choose the minimal $d \in \mathbb{N}$ and then the minimal $k \in \mathbb{N}_0$ such that $i \cdot a(ba)^k = i \cdot a(ba)^{k+d}$. Let

$$M = L \cap a\{1, ba, \ldots, (ba)^{k-1}\} \text{ and } N = L \cap a(ba)^k\{1, ba, \ldots, (ba)^{d-1}\}\ .$$

Then $\overline{M} \cup \overline{N}((b^+a^+)^d)^*$ is a “canonical” regular expression for the first summand of L.

We are interested in literal positive/boolean varieties consisting of literally idempotent languages. These varieties can be induced by classical varieties in two natural ways. At first, for a class of languages $\mathscr{V}$, we can consider the class of languages from $\mathscr{V}$ which are also literally idempotent languages, i.e. the intersection $\mathscr{V} \cap \mathscr{L}$.
The second possibility is to consider the following operator on classes of languages: $\mathscr{V} \mapsto \overline{\mathscr{V}}$ where

$$\overline{\mathscr{V}}(A) = \{ \overline{L} \mid L \in \mathscr{V}(A) \} \ .$$

The languages of the level 1/2 over A are exactly finite unions of languages of the form

$$A^*a_1A^*a_2\dots a_kA^*,\ k\in\mathbb{N}_0,\ a_1,\dots,a_k\in A\ . \tag{1/2}$$

We denote this positive variety of languages by $\mathscr{V}_{1/2}$.
The languages of the level 1 over A are exactly boolean combinations of languages of the form $(1/2)$. We denote this variety of languages by $\mathscr{V}_1$

Theorem (OK & LP)

(i) Finite unions of languages

$$A^* a_1 A^* a_2 \dots a_k A^*,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,\ a_1 \neq a_2 \neq \dots \neq a_k\ . \quad (\mathscr{L}\,1/2)$$

form a literal positive variety which is equal both to $\mathscr{V}_{1/2} \cap \mathscr{L}$ *and* $\overline{\mathscr{V}_{1/2}}$.

(ii) Finite unions of languages

$$B_1^* B_2^* \dots B_k^*,\ k \in \mathbb{N}_0,\ B_1, \dots, B_k \subseteq A\ . \quad (\mathscr{L}\,1/2\ c)$$

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(iii) Boolean combinations of languages of the form $(\mathscr{L}\,1/2)$ *form a literal boolean variety which is equal both to* $\mathscr{V}_1 \cap \mathscr{L}$ *and* $\overline{\mathscr{V}_1}$.

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In general, for a positive/boolean variety $\mathscr{V}$, the class $\mathscr{V} \cap \mathscr{L}$ is a literal positive/boolean variety but we have only $\mathscr{V} \cap \mathscr{L} \subseteq \overline{\mathscr{V}}$ and $\overline{\mathscr{V}}$ need not to be a literal positive/boolean variety.
For more, see O. Klíma and L. Polák, On varieties of literally idempotent languages, RAIRO - Theoretical Informatics and Applications Vol. 42 No. 3, p. 583 - 598

Two variable case

Let M_n be monoid with the presentation

$$< a_1, \ldots, a_n \mid a_1^2 = a_1, \ldots, a_n^2 = a_n >$$

and let $M_n = \{a_1, \ldots, a_n\}^* / \approx$.

Each $\pi \in \mathrm{Con} A^*$ with $\pi \supseteq \approx$ defines $\pi/\approx \in \mathrm{Con} M_n$ by $u \approx \ \pi/\approx \ v \approx$ iff $u \, \pi \, v$.

Theorem (OK & LP)

$\mathcal{V}(\{a_1, \ldots, a_n\})$'s for equational literal varieties of literally idempotent languages correspond to literally invariant congruences on M_n; we write $\mathcal{V}(\{a_1, \ldots, a_n\}) \mapsto \kappa_{\mathcal{V}(\{a_1, \ldots, a_n\})}$.
More precisely, $\kappa_{\mathcal{V}(\{a_1, \ldots, a_n\})}$ is the greatest literally invariant congruence on M_n containing all $\sim_L / \approx$, $L \in \mathcal{V}(\{a_1, \ldots, a_n\})$.
Moreover,
$L \in \mathcal{V}(\{a_1, \ldots, a_n\})$ if and only if $\sim_L / \approx \supseteq \kappa_{\mathcal{V}(\{a_1, \ldots, a_n\})}$.

From now on, let $n = 2$. We write $a = a_1,\ b = a_2$ an we identify the elements of M_2 with

$$1,\ u_{2\ell+1} = a(ba)^{\ell},\ u_{2\ell+2} = (ab)^{\ell+1},$$

$$v_{2\ell+1} = b(ab)^{\ell},\ v_{2\ell+2} = (ba)^{\ell+1},\ \ell \in \mathbb{N}_0\ .$$

On M_2 we have:
the trivial congruence $\Delta = \{\,(w,w) \in M_2 \times M_2 \mid w \in M_2\,\}$
and the universal congruence $\nabla = M_2 \times M_2$.

For $k \in \mathbb{N},\ d \in \mathbb{N}$, we put

$$U_{k,d} = \{u_k, u_{k+2d}, \dots\} \quad \text{and} \quad V_{k,d} = \{v_k, v_{k+2d}, \dots\}\ ;$$

and we write simply U_k instead of $U_{k,1}$ and V_k instead of $V_{k,1}$.

For $k, d \in \mathbb{N}$, consider the following equivalences on the set M_2:

- $\rho_{k,d}$ with non-trivial (= non-singleton) classes

$$U_{k,d},\ U_{k+1,d}, \ldots, U_{k+2d-1,d}, V_{k,d},\ V_{k+1,d}, \ldots, V_{k+2d-1,d}\ ,$$

- σ_k with the non-trivial classes $U_k \cup U_{k+1}$ and $V_k \cup V_{k+1}$,
- τ_k with the non-trivial classes $U_k \cup V_{k+1}$ and $U_{k+1} \cup V_k$,
- ν_k with the non-trivial class $U_k \cup U_{k+1} \cup V_k \cup V_{k+1}$.

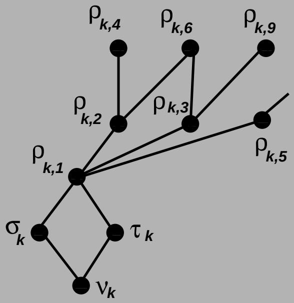
Theorem (OK & LP)

Proper literally invariant congruences of the monoid M_2 are exactly the relations listed above. They are generated by

$$u_k = u_{k+2d},\ u_k = u_{k+1},\ u_k = v_{k+1},\ u_k = v_k,\ \textit{respectively}\ .$$

The congruence ∇ is generated by $a = 1$.

A part of the dual of the k-th level of the lattice of all literally invariant congruences on M_2 is depicted below.



The whole lattice is the product of such a level with the chain $1 < 2 < \dots$ of all positive natural numbers with ∇ and Δ adjoined. We draw the dual since we are primarily interested in the corresponding varieties of languages.

Let L be a literally idempotent language over the alphabet $\{a,b\}$. Let $\mathscr{A} = (Q, \{a,b\}, \cdot, i, T)$ be the minimal complete deterministic automaton for L. We would like to find the smallest possible $\mathscr{V}(\{a,b\})$ containing L. So we are looking for the greatest literally invariant congruence on M_2 contained in $\sim_L$.

It is well-known that the syntactic homomorphism ϕ_L can be identified with the mapping which maps $u \in \{a,b\}^*$ onto the transformation of Q induced by the word u.

Recall that the automaton $\mathscr{A}$ is literally idempotent.

We distinguish several cases:

(i) There is the only cycle in $\mathscr{A}$ and it is of length 1.

(ii) There are exactly two cycles in $\mathscr{A}$, both of the length 1.

(iii) There is the only cycle in $\mathscr{A}$ and it is of length 2.

(iv) There is the only cycle in $\mathscr{A}$ and it is of length $2d$ where $d \geq 2$
or there are exactly two cycles of lengths $2d_1$ and $2d_2$ where $d_1, d_2 \in \mathbb{N}$
or there are exactly two cycles of lengths $2d$ and 1.

Notice that exactly one case of (i) – (iv) happens. Let k be the smallest such that all words w of length $\geq k$ transform the initial state i into a cycle. In the second subcase of (iv), let d equal to the least common multiple of d_1 and d_2.

Theorem (OK & LP)

The literally invariant congruence on M_2 corresponding to the language L is

ν_k *in the case (i),*

σ_k *in the case (ii),*

τ_k *in the case (iii),*

$\rho_{k,d}$ *in the case (iv).*

The last results are from
OK & LP, Literally idempotent languages and their varieties - two letter case, Proceedings AFL 2008.

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