M-hyperquasi-identities

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Definition

Let τ be a fixed type τ: $I \to N$, where I is an index set and N is the set of all natural numbers. Then F denotes the set of all fundamental operations $F = \{f_i : i \in I\}$ of type τ, i.e. $\tau(i)$ is the arity of the operation symbol f_i, for $i \in I$. Let $\sigma = (t_i : i \in I)$ be a fixed choice of terms of type τ with $\tau(t_i) = \tau(f_i)$, for every $i \in I$.

Definition

Recall that for a given σ, the extension of σ to the map $\overline{\sigma}$ from the set $T(\tau)$ to $T(\tau)$, leaving all the variables unchanged and acting on composed terms as:

$$\overline{\sigma}(f_i(p_0, ..., p_{n-1})) = \sigma(f_i)(\overline{\sigma}(p_0), ..., \overline{\sigma}(p_{n-1}))$$

is called a **hypersubstitution** of type τ.

In the sequel, we shall use σ instead of $\overline{\sigma}$ for a hypersubstitution. A hypersubstitution σ will be called **trivial**, if it is the identity mapping.

The set of all hypersubstitutions of type τ will be denoted by $H(\tau)$. Note, that $H(\tau)$ is a monoid, as a composition $\circ$ of two hypersubstitutions of a given type is a hypersubstitution.

Therefore:

$$\mathbf{H}(\tau) = (H(\tau), \circ)$$

denotes the **monoid of all hypersubstitutions** of type τ, with composition $\circ$ and the identity hypersubstitution as the unit.

Let a monoid $\mathbf{M} = (M, \circ)$ of hypersubstitutions of a given type τ be given.

Derived varieties were invented by P. Cohn [1].
Derived varieties of a given type were invented by EG and DS in [4]. In [5] we introduced the notion of a *M-hyper(quasi)-identity* and *M-hyper(quasi)variety*

For any algebra $\mathbf{A} = (A, \Omega) = (A, (f_i^{\mathbf{A}} : i \in I)) \in V$, of type τ, the algebra $\mathbf{A}_\sigma = (A, (t_i^{\mathbf{A}} : i \in I))$ or shortly $\mathbf{A}_\sigma = (A, \Omega_\sigma)$, for $\Omega_\sigma = (t_i : i \in I)$ is called a **derived algebra** (of a given type τ) of $\mathbf{A}$, corresponding to σ, for any $\sigma \in H(\tau)$.
If $\sigma \in M$, then a derived algebra $\mathbf{A}$ is called an M-**derived**.

Definition

$D_M(K)$ denotes the class of all M-derived algebras of K for all possible choices of $\sigma \in M$ of type τ, i.e.:

$$D_M(K) = \bigcup\{\mathbf{A}_\sigma : \mathbf{A} \in K, \sigma \in M\}.$$

$D_M = D$, for $M = H(\tau)$ is a class operator examined by EG and DS in [4].

Definition

For a given set Σ of identities of type τ, $E(\Sigma)$ denotes the set of all **consequences** of Σ by the *rules* (1)-(5) of **inferences** of G. Birkhoff.

$Mod(\Sigma)$ denotes the variety of algebras determined by Σ.

A variety V is **trivial** if all algebras in V are **trivial** (i.e. one-element). Trivial varieties will be denoted by T. A subclass W of a variety V which is also a variety is called **subvariety** of V. V is **minimal** (or **equationally complete**) variety if V is not trivial but the only subvariety of V, which is not equal to V is trivial.

Definition

A (quasi)variety V of type τ is M-**solid** if V contains all M-derived algebras of V, i.e. $D_M(V) \subseteq V$.

We recall only the definitions of the fact that a hyperidentity is satisfied in an algebra as an M-hyperidentity of a given type and the notion of M-hypervariety:

Definition

An algebra **A satisfies** a hyperidentity $p \approx q$ as **M-hyperidentity** if for every M-hypersubstitution $\sigma \in M$ of the hypervariables by terms (of the same arity) of **A** leaving the variables unchanged, the identities which arise hold in **A**.
In this case, we write

$$\mathbf{A} \models_H^M (p \approx q).$$

A variety V **satisfies** a hyperidentity $p \approx q$ as **M-hyperidentity** if every algebra in the variety does.
In that case, we write:

$$V \models_H^M (p \approx q).$$

Definition

A class V of a algebras of a given type is called an **M-hypervariety** if and only if V is defined by a set M-hyperidentities.

We got the following:

Theorem

A variety V of type τ is defined by a set Σ of M-hyperidentities if and only if $V = HSPD_M(V)$, i.e. V is a variety closed under M-derived algebras of type τ. Moreover, in this case, the set Σ is then M-hypersatisfied in V and V is the class of all M-hypermodels of Σ, i.e. $V = MHMod(\Sigma)$.

Recall the classical definition of Mal'cev:

Definition

A quasi-identity e is an implication of the form:

$$(t_0 \approx s_0) \wedge \dots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n),$$

where $t_i \approx s_i$ are k-ary identities of a given type, for $i = 0, \dots, n$.

A quasi-identity is **satisfied in an algebra A** of a given type if and only if the following implication is satisfied in **A**: given a sequence $a_1, ..., a_k$ of elements of A. If this elements satisfy the equations $t_i(a_1, ..., a_k) = s_i(a_1, ..., a_k)$ in **A**, for $i = 0, 1, ..., n-1$, then the equality $t_n(a_1, ..., a_k) = s_n(a_1, ..., a_k)$ is satisfied in **A**.
In that case we write:

$$\mathbf{A} \models (t_0 \approx s_0) \wedge ... \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

Definition

A quasi-identity e is **satisfied in a class** V of algebras of a given type, if and only if it is satisfied in all algebras $\mathbf{A}$ belonging to V. In that case we write:

$$V \models (t_0 \approx s_0) \wedge \dots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

Definition

A hyperquasi-identity e is formally the same as a quasi-identity.

A hyperquasi-identity e is **M-hyper-satisfied** (holds) in an algebra **A** if and only if the following implication is satisfied:
If σ is a hypersubstitution of M and the elements $a_1, ..., a_k \in A$ satisfy the equalities $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in **A**, for $i = 0, 1, ..., n-1$, then the equality

$$\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k) \text{ holds in } \mathbf{A}.$$

We say then, that e is an **M-hyperquasi-identity** of **A** and write:

$$\mathbf{A} \models_H^M (t_0 \approx s_0) \wedge \ldots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n).$$

Definition

A hyperquasi-identity e is **M-hyper-satisfied** (**holds**) in a class V if and only if it is M-hypersatisfied in any algebra of V.

By other words, M-hyperquasi-identity is a universally closed Horn $\forall x \forall \sigma$-formulas, where x vary over all sequences of individual variables (occurring in terms of the implication) and σ vary over all hypersubstitutions of M.

Remark

All hyperquasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn $\forall$-formulas. In case of a trivial monoid M, the notion of M-hypersatisfaction reduces to the notion of classical satisfaction. If M is the monoid of all hypersubstiutions of a given type τ, then the notion of M-hyperidentity and M-hyperquasi-identity reduces to the hyperidentity and hyperquasi-identity.

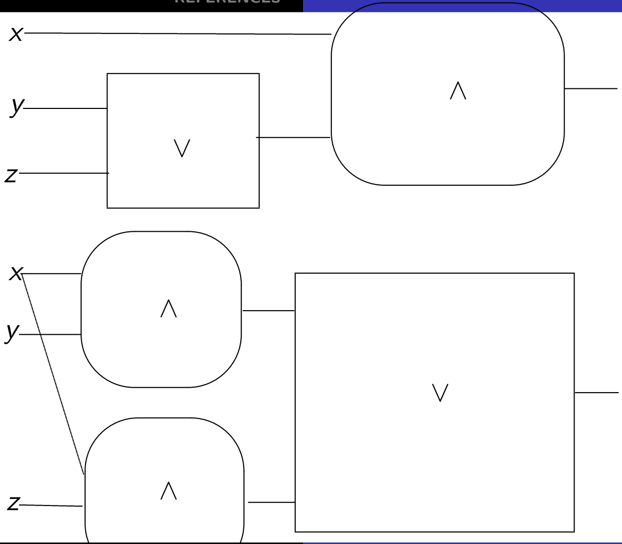
A lattice $\mathbf{L} = (L, \wedge, \vee)$ is *distributive* if the following two *distributive laws* hold in $\mathbf{L}$, called $\wedge$-distributivity and $\vee$-distributivity, respectively:

$$(D_\wedge)\ x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z),$$
$$(D_\vee)\ x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

Definition

For a lattice $\mathbf{L} = (L, \wedge, \vee)$ the lattice $\mathbf{L}^d = (L, \vee, \wedge)$ is called the *dual lattice*.
For a variety V of lattices, the variety V^d consisting of all dual lattices of lattices of V, is called the *dual variety* of V.
If $V = V^d$, then V is called *selfdual*.

All axioms of (distributive) lattices are mutually dual. Therefore if a lattice is distributive, then its dual is distributive. This fact is well known as *duality principle* in the variety of (distributive) lattices. It is well known that these distributive laws may be expressed as two pairs of equivalent switching circuits: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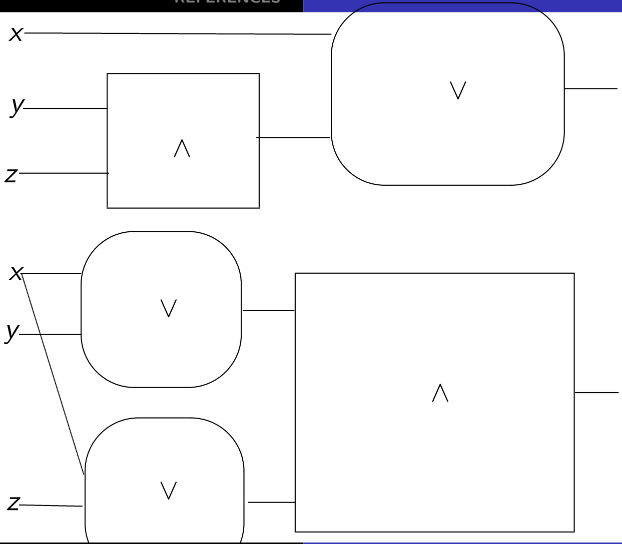
And dually: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The notion of a hyperidentity may be regarded as a kind of a generalization of the principle of duality in (distributive or modular) lattices. This fact was first noticed by D. Schweigert in Lemma 10 of [7]. Here we use the notation of a *hyperidentity* [4]:

Theorem

Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice. Then $\mathbf{L}$ is a distributive lattice if and only if the distributivity law $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ and the dual are satisfied in $\mathbf{L}$ as hyperidentities of type (2,2), i.e. if the following hyperidentity:

$$F(x, G(y, z)) \approx G(F(x, y), F(x, z))$$

is satisfied as a hyperidentity in $\mathbf{L}$.

Proof

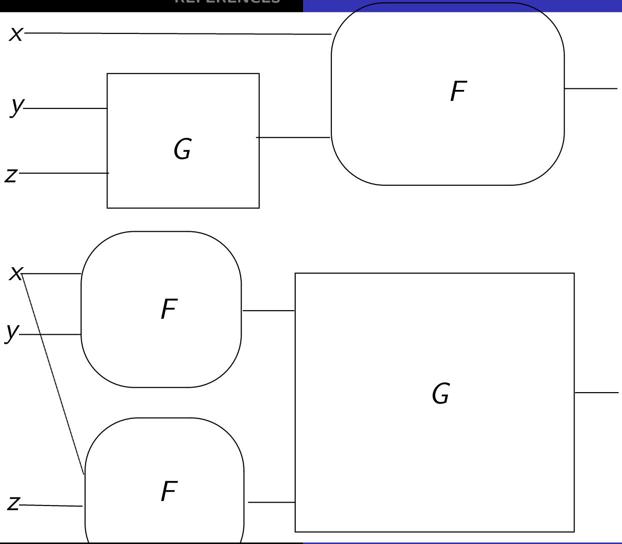
To prove the necessity, let us consider all possible nonequivalent hypersubstitutions (i.e. all possible hypersubstitutions of binary lattice terms) using a technique described by J. Płonka:
Case 1. $F(x, y) := x$.
Obviously $x \approx G(x, x)$ is satisfied in **L**.
Case 2. $F(x, y) := y$. Then the following identity is satisfied in **L**:
$G(y, z) \approx G(y, z)$.

Case 3. $F(x,y) := x \wedge y$. Consider $G(y,z) := y, z, y \wedge z, y \vee z$. Then the following identities hold in $\mathbf{L}$:
$x \wedge y \approx x \wedge y$, $x \wedge z \approx x \wedge z$, $x \wedge (y \wedge z) \approx (x \wedge y) \wedge (x \wedge z)$, $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$. The last one follows from the distributivity of $\mathbf{L}$.
Case 4. $F(x,y) := x \vee y$. Consider $G(y,z) := y, z, y \wedge z, y \vee z$. Then the following identities hold in $\mathbf{L}$:
$(x \vee y) \approx (x \vee y)$, $(x \vee z) \approx (x \vee z)$,
$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$, $x \vee (y \vee z) \approx (x \vee y) \vee (x \vee z)$.
The last but one one follows from the distributivity of $\mathbf{L}$.
For sufficiency, note that satisfaction of the hyperidentity above in a lattice $\mathbf{L}$ implies the distributivity of $\mathbf{L}$. $\square$

Remark

Therefore distributive lattices may be described as lattices which satisfy the distributive law as a hyperidentity of type (2,2).

The fact above may be visualized by the following pair of equivalent circuits, where two kinds of binary boxes express a possibility of a substitution of any binary lattice terms F and G instead of them:


x
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F
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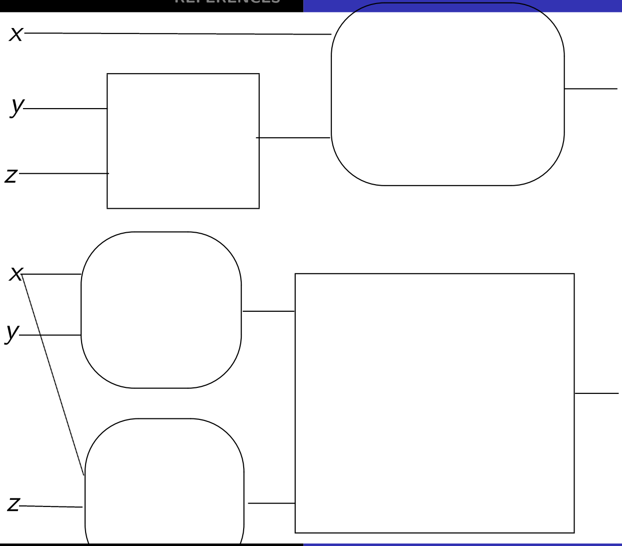
F
F
G

Or in an equivalent way:


x
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Definition

A lattice $\mathbf{L} = (L, \wedge, \vee)$ is *modular* if the following *modular law* $(M_\vee)$ holds in $\mathbf{L}$:

$$(M_\vee)\ (x \wedge y) \vee (x \wedge z) \approx x \wedge (y \vee (x \wedge z)).$$

If a lattice is modular, then its *dual lattice* is modular as well, therefore the following dual law is valid in modular lattices:

$$(M_\wedge)\ (x \vee y) \wedge (x \vee z) \approx x \vee (y \wedge (x \vee z)).$$

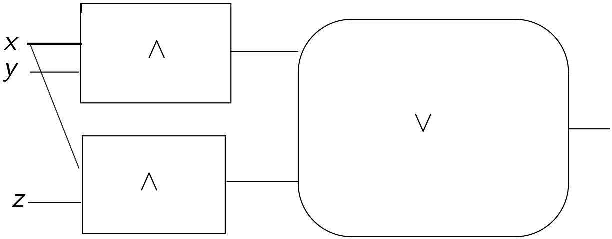
It is well known, that modularity of a lattice $\mathbf{L} = (L, \wedge, \vee)$ may be expressed by the following implication:

$$x \geq z \rightarrow (x \wedge y) \vee z \approx x \wedge (y \vee z),$$

or equivalently, by the following quasi-identity of type (2,2):

$$x \vee z \approx x \rightarrow (x \wedge y) \vee z \approx x \wedge (y \vee z).$$

The modularity law $(M_\vee)$ in a lattice **L** may be expressed as a pair of equivalent switching circuits (or the pair of theirs duals):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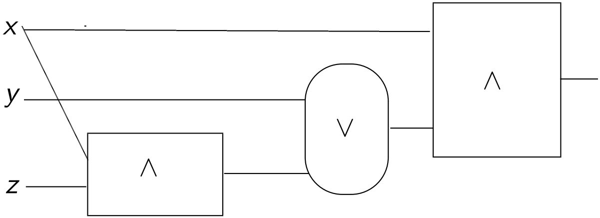
x
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Similarly as in the case of distributivity one may show that the modularity law is satisfied in the variety of lattices as a hyperidentity. In fact, the following is true:

Theorem

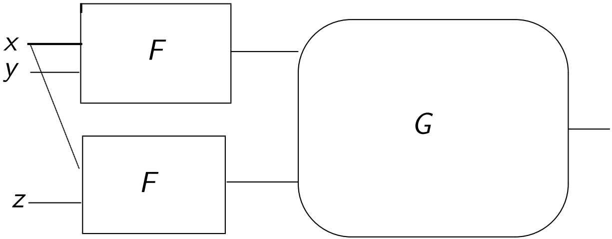
A lattice $\mathbf{L} = (L, \wedge, \vee)$ is modular if and only if the following hyperidentity of type (2,2) of modularity is satisfied in $\mathbf{L}$:

$$G(F(x,y), F(x,z)) \approx F(x, G(y, F(x,z))).$$

Proof.

For a proof, consider all non-equivalent hypersubstitutions of binary hyperterms: $F(x,y)$ and $G(x,y)$ by x, y, $x \wedge y$ and $x \vee y$, respectively.

The fact above may be visualized by a pair of equivalent switching circuits (in a modular lattice **L**):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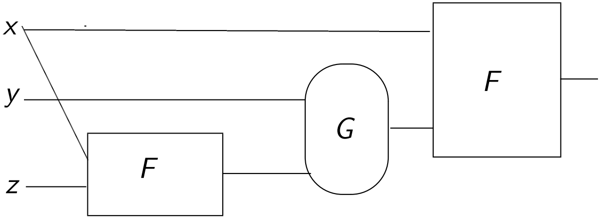
y
z
F
F
G


x
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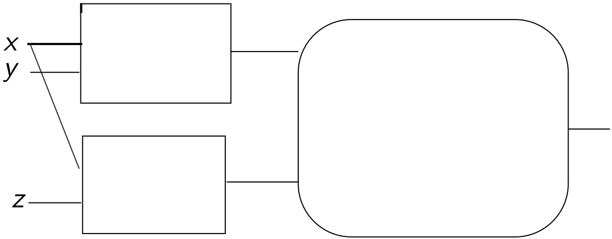
F
G
F

Or in an equivalent form:


x
y
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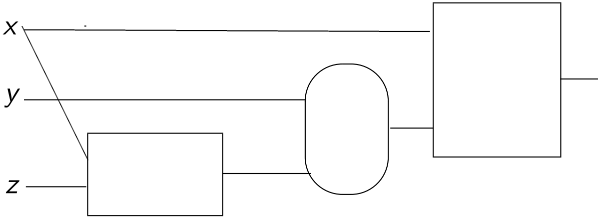
x
y
z

Recall the definition of a *hyperquasi-identity* of a given type τ from [2], [5]:

Definition

A quasi-identity e: $p \approx q \rightarrow r \approx s$ of type τ is satisfied in an algebra $\mathbf{A}$ of type τ as a hyperquasi-identity if and only if for every $\sigma \in H(\tau)$, the derived quasi-identity $\sigma(e)$:

$$\sigma(p) \approx \sigma(q) \rightarrow \sigma(r) \rightarrow \sigma(s)$$

is satisfied in $\mathbf{A}$.

Modularity may be expressed by the following hiperquasi-identity:

Theorem

A lattice $\mathbf{L} = (L, \wedge, \vee)$ is modular if and only if the following hyperquasi-identity of type (2,2) of modularity is satisfied in $\mathbf{L}$:

$$G(x, z) \approx x \rightarrow G(F(x, y), z) \approx F(x, G(y, z)).$$

A lattice is **joinsemidistributive** if it satisfies the following condition:

$$(SD_\vee)\ x \vee y \approx x \vee z \rightarrow x \vee y \approx x \vee (y \wedge z)$$

The **meetsemidistributivity** is defined by duality:

$$(SD_\wedge)\ x \wedge y \approx x \wedge z \rightarrow x \wedge y \approx x \wedge (y \vee z)$$

A lattice is **semidistributive** if it is simultaneously join and meet semidistributive.

The following may be regarded as a characterization* of semidistributivity by a hyperquasi-identity:

Proposition

Let $\mathbf{L} = (L, \wedge, \vee)$ *be a lattice. Then* $\mathbf{L}$ *is semidistributive if and only if the following hyper quasi-identity is hypersatisfied in* $\mathbf{L}$*:*

$$(F(x, y) \approx F(x, z)) \rightarrow (F(x, y) \approx F(x, G(y, z)))$$

In [4] we used the name M-hyperquasivariety for M-solid quasivarieties. In fact the following holds:

Theorem

A class K of algebras of a given type is an M-hyper(quasi)variety if and only if it is an M-solid (quasi)variety.

We got the following reformulation of Mal'cev classical result:

Theorem

A class K of algebras of a given type is M-hyperquasivariety (M-solid quasivariety) if and only if K:
i) is ultraclosed;
ii) is heraditery;
iii) is multiplicatively closed;
iv) contains a trivial system;
v) is M-deriverably closed.

In this section we present a solution of the following particular case of the Problem 32 of K. Denecke and S.L. Wismath [2]:
(P.32) *Give the derivation rules for M-hyperquasi-equational logic*.

First, we shall consider the case where the monoid M is trivial, i.e. $M = H(\tau)$. In the sequel, **E** denotes the *equational logic*, i.e. the fragment of the first-order logic without relation symbols. The formulas of **E** are all possible identities of a given type τ, the set of axioms Eq of **E** are identities of the form $p \approx p$, and the rules of inferences are the equality rules (atomic formulas are regarded as identities) and the *substitution rule*, i.e. G. Birkhoff's rules (1)–(5) of derivation.

E denotes the set of equality axioms of a given type τ.

For a set Σ of (hyper)quasi-identities of a given type τ, $HQMod(\Sigma)$ denotes the class of all algebras **A** which hypersatisfy all elements of Σ.

HE denotes the *hyperequational logic*, i.e. the fragment of the second-order logic, extending the equational logic. The formulas and axioms are the same as in **E**. To the inference rules we add the rule (6) of hypersubstitution defined in by the authors of [4].

A quasi-identity e is called a *consequence* of the set Σ of quasi-identities if for every algebra $\mathbf{A}$ of type τ,

$$\mathbf{A} \models \Sigma \text{ implies that } \mathbf{A} \models e.$$

In symbols:

$$\Sigma \models e.$$

We say that an identity e is a *hyperconsequence* of a set of quasi-identities Σ, if for every algebra $\mathbf{A} \in HMod(\Sigma)$, it follows that $\mathbf{A} \models_H e$, i.e.:

$$\mathbf{A} \models_H \Sigma \text{ implies } \mathbf{A} \models_H e.$$

In symbols:

$$\Sigma \models_H e.$$

We use the following notation:

$\Delta \rightarrow \alpha$, for a set $\Delta = \{p_i \approx q_i : 0 \leq i \leq n-1\}$ and $\alpha = p_n \approx q_n$

instead of the quasi-identity:

$$p_0 \approx q_0 \wedge ... \wedge p_{n-1} \approx q_{n-1} \rightarrow p_n \approx q_n.$$

We adopt the convention, that an identity $p \approx q$ may be regarded as a quasi-identity e of the form $\emptyset \rightarrow p \approx q$, where $\emptyset$ denotes the empty set.

G. Birkhoff's well known theorem is called *the completeness theorem*:

Theorem

An identity e is a consequence of a set Σ of identities if and only if e is derived from Σ in $\mathbf{E}$.

The question naturally arises of when an identity is a consequence of a set of quasi-identities Σ. Following V.A.G. [2] it is necessary, together with a substitution rule to consider the *modus ponens* rule:

$$\text{(MP)} \quad \frac{\alpha, \{\alpha\} \cup \Delta \to \beta}{\Delta \to \beta}.$$

Recall, that in the quasi-equational logic $\mathbf{Q}$ (of a given type τ), without relation symbols, the formulas are all possible quasi-identities of a given type τ, the axioms are the *equality axioms* (E.1) – (E.4) and the inference rules are the *substitution rule*, the *cut rule* and the *extension rule*. We list all of them.

Axioms:

(E.1) the reflexivity:

$$\emptyset \to p \approx p,$$

(E.2) the symmetry:

$$p \approx q \to q \approx p,$$

(E.3) the transitivity:

$$(p \approx q) \wedge (q \approx r) \to (p \approx r),$$

or in an equivalent notation:

$$\{p \approx q, q \approx r\} \to p \approx r,$$

(E.4) the compatibility:

$$(t_0 \approx s_0) \wedge \ldots \wedge (t_{\tau(f)-1} \approx s_{\tau(f)-1}) \longrightarrow (f(t_0, \ldots, t_{\tau(f)-1}) \approx f(s_0, \ldots, s_{\tau(f)-1})),$$

for every operation symbol f of type τ,
or in an equivalent notation:

$$\{t_0 \approx s_0, \ldots, t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \longrightarrow (f(t_0, \ldots, t_{\tau(f)-1}) \approx f(s_0, \ldots, s_{\tau(f)-1})),$$

for every operation symbol f of type τ.

The inference rules are the following rules:
(8.1) the substitution rule (where δ is a substitution of variables):

$$\frac{\{\gamma_0,\ldots,\gamma_{n-1}\}\rightarrow\beta}{\{\delta(\gamma_0),\ldots,\delta(\gamma_{n-1})\}\rightarrow\delta(\beta)}$$

(8.2) the cut rule:

$$\frac{\Delta\rightarrow\alpha,\{\alpha\}\cup\Gamma\rightarrow\beta}{\Delta\cup\Gamma\rightarrow\beta}$$

(8.3) the extension rule:

$$\frac{\Delta\rightarrow\alpha}{\{\beta\}\cup\Delta\rightarrow\alpha}.$$

We write $\Sigma \vdash_Q e$ if there exists a derivation of a quasi-identity e from a set Σ of quasi-identities in $\mathbf{Q}$.

The classical result by many authors is the following:

Theorem

A quasi-identity e is a consequence of a set Σ of quasi-identities if and only if e is derivable from Σ in $\mathbf{Q}$.
In symbols:

$$\Sigma \models_Q e \text{ if and only if } \Sigma \vdash_Q e.$$

We modify *quasi equational logic* **Q** to *hyperquasi-equational logic* **HQ** by adding a new rule:
(8.4) a hypersubstitution rule (where σ is a hypersubstitution of $H(\tau)$):

$$\frac{(t_0 \approx s_0) \wedge \ldots \wedge (t_{n-1} \approx s_{n-1}) \rightarrow (t_n \approx s_n)}{\sigma(t_0) \approx \sigma(s_0) \wedge \ldots \wedge \sigma(t_{n-1}) \approx \sigma(s_{n-1}) \rightarrow \sigma(t_n) \approx \sigma(s_n)},$$

or in an equivalent notation:
(8.4) a hypersubstitution rule (where σ is a hypersubstitution of $H(\tau)$):

$$\frac{\{\gamma_0,\ldots,\gamma_{n-1}\}\rightarrow\beta}{\{\sigma(\gamma_0),\ldots,\sigma(\gamma_{n-1})\}\rightarrow\sigma(\beta)}.$$

Definition

By **HQ** we denote the hyperquasi-equational logic, which is an extension of the hyperequational logic **HE** based on the equality axioms E and four rules (8.1) – (8.4) above.

We write $\Sigma \vdash_{HQ} e$ if there exists a derivation of e from Σ in **HQ**. We write $\Sigma \models_{HQ} e$ if e is a hyperconsequence of Σ, considered as a hyperbase, i.e. if $\mathbf{A} \in HQMod(\Sigma)$, then $\mathbf{A} \models_{HQ} e$.

Definition

A set Σ of quasi-identities of type τ is called *hyperclosed* if and only if it is closed under the equality axioms, the substitution rule, hypersubstitution rule, the cut rule, the extension rule.

We reformulate the classical results in the following way:

Theorem

A set Σ is a set of all (hyper)quasi-identities of type τ, (hyper)satisfied in a class $\mathcal{K}$ of algebras of type τ if and only if it is (hyper)closed.

Proof

If Σ is a set of all hyperquasi-identities hypersatisfied in a class $\mathcal{K}$ of algebras of type τ, then it is closed in $\mathbf{Q}$, i.e. is closed under the equality axioms and the substitution rule, the cut and the extension rule. In consequence it is also closed under the rules of equational logic. If e is a quasi-identity of Σ, then for every $\sigma \in H(\tau)$, the hypersubstitution $\sigma(e)$ of e by σ is satisfied in $\mathcal{K}$. Therefore Σ is closed under the hypersubstitution rule (8.4). In case if e is an identity of type τ, we conclude that $\sigma(e)$ is satisfied in $\mathcal{K}$ for every $\sigma \in H(\tau)$. Therefore Σ is closed under the rule (6) of hypersubstitution, i.e. is hyperclosed.

Assume now, that Σ is hyperclosed. Therefore it is closed. We conclude that Σ is a set of quasi-identities satisfied in a class $\mathcal{K}$ of algebras of type τ. As Σ is hyperclosed, therefore for every quasi-identity e of Σ and every $\sigma \in H(\tau)$, the derived quasi-identity $\sigma(e)$ is also satisfied by $\mathcal{K}$, which means that $\mathcal{K}$ is a class of algebras of type τ, which hypersatisfies Σ. $\square$

The clue of the next proofs is the following:

Proposition

A derivation from Σ in **HQ** *means a derivation from $(8.4)(\Sigma)$ in* **Q***, i.e. one first need to close the set Σ under the hypersubstitution rule (8.4) and then under the equality axioms and other rules. The resulting set will be already closed under all axioms and inference rules of* **HQ***.*

More precisely:

Proposition

The hypersubstitution rule (8.4) commutes with all the axioms and rules of the logic **HQ***.*

Proof

First, we note that the assertion easily holds the equality axioms (E1)–(E3). Moreover, by an easy induction on the complexity of terms, the following generalization of the rule (E.4) is valid in the logic **Q**:

$$\text{(GE.4)}\ \{t_0 \approx s_0, \dots, t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \rightarrow (p(t_0, \dots, t_{\tau(f)-1}) \approx p(s_0, \dots, s_{\tau(f)-1})),$$

for every term p of type τ.

We prove that if the axiom (E.4) is applied first:

$$\{t_0 \approx s_0, \ldots, t_{\tau(f)-1} \approx s_{\tau(f)-1}\} \rightarrow (f(t_0, \ldots, t_{\tau(f)-1}) \approx f(s_0, \ldots, s_{\tau(f)-1})),$$

and then the hypersubstitution rule (8.4) is applied by a hypersubstitution σ:

$$\{\sigma(t_0) \approx \sigma(s_0), ..., \sigma(t_{\tau(f)-1}) \approx \sigma(s_{\tau(f)-1})\} \longrightarrow$$
$$(\sigma(f(t_0, ..., t_{\tau(f)-1})) \approx \sigma(f(s_0), ..., (s_{\tau(f)-1}))),$$

then one may apply rule (GE.4) with $p = \sigma(f(x_0, ..., x_n))$, to obtain the resulting quasi-identity:
$\{\sigma(t_0) \approx \sigma(s_0), ..., \sigma(t_{\tau(f)-1}) \approx \sigma(s_{\tau(f)-1})\} \longrightarrow$
$(\sigma(f)(\sigma(t_0), ..., \sigma(t_{\tau(f)-1})) \approx \sigma(f)(\sigma(s_0), ..., \sigma(s_{\tau(f)-1}))).$

Now we prove the assertion for the modus ponens rule (MP):

$$\text{(MP)}\ \frac{\alpha, \{\alpha\} \cup \Delta \rightarrow \beta}{\Delta \rightarrow \beta}.$$

i.e. we will show, that if the (MP) rule is applied first and then the hypersubstitution rule (8.4) is applied to deduce a quasi-identity $e = \sigma(\Delta) \rightarrow \sigma(\beta)$, then one may apply the hypersubstitution rule (8.4) first to α and $\alpha \cup \Delta \rightarrow \beta$ and then (MP), which leads to the quasi-identity e as well.

Assume naw that the substitution rule (8.1) is applied (where δ is a substitution of variables):

$$(8.1)\ \frac{\{\gamma_0,\ldots,\gamma_{n-1}\}\rightarrow\beta}{\{\delta(\gamma_0),\ldots,\delta(\gamma_{n-1})\}\rightarrow\delta(\beta)}$$

and then the hypersubstitution rule (8.4) is applied to get the quasi-identity:

$$(*)\ \{\sigma(\delta(\gamma_0)),\ldots,\sigma(\delta(\gamma_{n-1}))\}\rightarrow\sigma(\delta(\beta))$$

for some hypersubstitution $\sigma\in H(\tau)$ and a substitution δ of variables.

Assume that the substitution δ acts on variables $x_0, ..., x_m$ of terms $\gamma_0, ..., \gamma_{n-1}, \beta$ putting: $\delta(x_k) = p_k$, then putting $\delta_1(x_k) = \sigma(p_k)$ on variables of terms $\sigma(p_k)$ of type τ, we get that:
$\sigma(\delta_1(\gamma_i)) = \delta_1(\sigma(\gamma_i))$, for $i = 0, ..., n-1$ and
$\sigma(\delta_1(\beta)) = \delta_1(\sigma(\beta))$.
We conclude that the quasi-identity (*) equals to the quasi-identity:

$$(*)\ \{\delta_1(\sigma(\gamma_0)), ..., \delta_1(\sigma(\gamma_{n-1}))\} \rightarrow \delta_1(\sigma(\beta)),$$

which means that one may apply the hypersubstitution rule (8.4) first and then the substitution rule (8.1) to get the same result.

The proof for the cut rule is similar. Assume that the cut rule (8.2) is applied:

$$(8.2)\ \frac{\Delta \to \alpha, \{\alpha\} \cup \Gamma \to \beta}{\Delta \cup \Gamma \to \beta}$$

and then the hypersubstitution rule (8.4) by a hypersubstitution σ gives rise to the quasi-identity:

$$(**)\ \sigma(\Delta) \cup \sigma(\Gamma) \to \sigma(\beta).$$

Then one may apply the hypersubstitution rule (8.4) by σ to the quasi-identities:

$$\Delta \to \alpha \text{ and } \{\alpha\} \cup \Gamma \to \beta$$

to get the resulting quasi-identity (**) via the cut rule (8.2).

We finalize with the proof of the statement for the extension rule, applying first:

$$(8.3)\ \frac{\Delta \to \alpha}{\{\beta\} \cup \Delta \to \alpha}$$

and assuming that the hypersubstitution rule (8.3) by σ was applied then, leading to the quasi-identity:

$$(***)\ \{\sigma(\beta)\} \cup \sigma(\Delta) \to \sigma(\alpha).$$

Then one may apply the hypersubstitution rule (8.4) σ first to the quasi-identity: $\Delta \to \alpha$, to get the resulting quasi-identity (***) as a result of the extension rule (8.3). $\square$

The observation above is a generalization of that which has been already noticed in 1989, for the fact that derivation rules (1)-(5) of G. Birkhoff and the new rule (6) of hypersubstitution behave similarly, i.e. closing a set Σ of identities under (1)-(6) means, to close Σ under (6) first and then under rules (1)-(5) and we are done.
Therefore, we can say that the hyperquasi-equational logic is the one-step extension of the quasi-equational logic by the hypersubstitution rule (8.4).

We obtain a slight generalization of Corollary 2.2.3 of V.A.G. [3]:

Proposition

An identity e is a (hyper)consequence of a set Σ of quasi-identities if and only if there is a derivation of e (of $\sigma(e)$, for every $\sigma \in H(\tau)$) from $E \cup \Sigma$ by the substitution rule and modus ponens rule (and the hypersubstitution rule (8.4)).

The following is the modification of the classical *completeness theorem* of the logic **Q**:

Theorem

A (hyper)quasi-identity e is a (hyper)consequence of a set Σ *of (hyper)quasi-identities if and only if it is derivable from* Σ *in* **(H)Q***.*

In symbols: $\Sigma \models_{(H)Q} e$ *if and only if* $\Sigma \vdash_{(H)Q} e$*.*

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THANK YOU!