MV-pairs and states

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MV-algebra

MV-algebras were introduced by Chang in 1958; algebraic basis for many-valued logic. Equivalent definition by Mangani 1973:

An MV-algebra $(M; \oplus, *, 0)$ is a $(2, 1, 0)$ type of algebra:

(MV1) the binary operation $\oplus$ is commutative and associative with the nullary operation 0 as neutral element;

(MV2) $a \oplus 1 = 1$ where $1 = 0^*$;

(MV3) $1^* = 0$;

(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

(MV4) is called *Łukasiewicz axiom*. It guarantees that MV-algebra is a distributive lattice according to the ordering given by

$$a \leq b \text{ iff } a^* \oplus b = 1.$$

Boolean algebras coincide with MV-algebras satisfying the additional condition $x \oplus x = x$.
Mundici 1986: MV-algebras are in categorical equivalence with $[0, u]$ in Abelian lattice-ordered groups with strong unit u. A prototypical model of MV-algebra is the real unit interval $[0, 1]$.

MV-effect algebras

Chovanec and Kôpka: MV-algebras are a subclass of a more general algebraic structures called *effect algebras*.
An *effect algebra* (Foulis and Bennett 1994; Kôpka and Chovanec 1994):a partial algebraic structure $(E;+,0,1)$ with a partial binary operation $+$ satisfying
(1) $a+b=b+a$; $(a+b)+c=a+(b+c)$ (in the sense that if one side exists so does the other and equality holds);
(2) to every $a \in E$, there is a unique element a' such that $a+a'=1$;
(3) $a+1$ is defined $\implies a=0$.
Define $a \perp b$ iff $a+b$ is defined.
Partial order: $a \leq b$ if there is $c \in E$ with $a+c=b$. Put $c=b-a$.

We may define a partial operation $+$ on MV-algebra by restriction of $\oplus$ to the pairs of elements for which $a \leq b^*$. Then the structure $(M, +, 0, 1)$ is a lattice ordered effect algebra that satisfies RDP:

$$a \leq b + c \implies \exists b_1 \leq b, c_1 \leq c : a = b_1 + c_1.$$

Definition 1. *An MV-effect algebra is a lattice ordered effect algebra satisfying the Riesz decomposition property.*

An MV-effect algebra can be made an MV-algebra by putting $a \oplus b = a + a' \wedge b$ and $a^* = a'$.
There is a one-to-one correspondence between MV-effect algebras and MV-algebras.

Morphisms

Let E and F be effect algebras. A mapping $\phi : E \longrightarrow F$ is a *morphism* (of effect algebras) if:
(i) $\phi(1) = 1$;
(ii) If $a + b$ is defined then $\phi(a) + \phi(b)$ is defined and $\phi(a + b) = \phi(a) + \phi(b)$.
A morphism ϕ is *full* if whenever $\phi(a) + \phi(b) \in \phi(E)$, there are a_1 and b_1 in E such that $\phi(a) = \phi(a_1)$, $\phi(b) = \phi(b_1)$ and $a_1 \perp b_1$.
A bijective and full morphism is an isomorphism.

Congruences

A relation $\sim$ on an effect algebra is called a *congruence* if:
(C1) $\sim$ is an equivalence relation;
(C2) $a \sim a_1$, $b \sim b_1$ and $a \perp b, a_1 \perp b_1$ imply $a + b \sim a_1 + b_1$;
(C3) $a \sim b$, $c \perp b$ implies that there is d, $d \sim c$ and $d \perp a$.
We write $[a] = \{b \in E : a \sim b\}$ for the equivalence class of $a \in E$, and the set of all equivalence classes $E/\sim$ is organized into an effect algebra by defining $[a] \perp [b]$ if there are $a_1 \in [a], b_1 \in [b]$ with $a_1 \perp b_1$, and then putting $[a] + [b] = [a_1 + b_1]$.

Ideals

A subset I of an effect algebra E is an *ideal* if

$$a, b \in E,\ a \perp b \implies a + b \in I \text{ iff } a \in I, b \in I.$$

Define $a \sim_I b$ if $\exists\, i, j \in I : a - i = b - j$.
If E satisfies RDP, then for every ideal I, $\sim_I$ is a congruence.
M– MV-effect algebra.
(1) Every effect algebra ideal I is MV-ideal, and $M/\sim_I$ is an MV-effect algebra.
(2) If $\sim$ is an effect algebra congruence not generated by an ideal, then $M/\sim$ need not be MV-effect algebra.
(3) Every effect algebra congruence preserves RDP.

R-generated Boolean algebras

R-generated Boolean algebra $B(M)$: MV-algebra M as a distributive lattice generates $B(M)$ as a Boolean algebra and is its 0,1-sublattice. $B(M)$ is unique, up to isomorphism.

M*-chain representation* $\forall x \in B(M)$, :
$x = x_1 + \ldots + x_n$ where $x_i \in M$ for every $i \in \{1, \ldots n\}$, $x_1 \leq \ldots \leq x_n$ and $+$ denotes the symmetric difference in the Boolean algebra.
We may choose it such that every element will have an M-chain of even length.

Theorem 2. (Jenča,2004) *Let M be an MV-effect algebra. The mapping $\psi_M : B(M) \to M$ given by*

$$\psi_M(x) = \bigoplus_{i=1}^{n}(x_{2i} \ominus x_{2i-1}),$$

where $\{x_i\}_{i=1}^{2n}$ is an M-chain representation of x, is a surjective and full morphism of effect algebras.

Example 3. If M is linearly ordered MV-algebra, then $B(M)$ is isomorphic to the Boolean algebra of the subsets of M of the form $[a_1, b_1)\dot{\cup}\ldots\dot{\cup}[a_n, b_n)$. Then $\psi_M([a_1, b_1)\dot{\cup}\ldots\dot{\cup}[a_n, b_n)) = (b_1 \ominus a_1) \oplus \cdots \oplus (b_n \ominus a_n)$.

MV-pair

A *BG-pair* is a pair (B, G), where B is a Boolean algebra and G is a subgroup of the group of automorphisms of B. Define $a \sim_G b$ iff there exists $f \in G : a = f(b)$.

Definition 4. (Jenča,2007) *A BG-pair (B, G) is called an MV-pair iff the following conditions are satisfied:*
(MVP1) *For all $a, b \in B$, $f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a) = f(a)$ and $h(b) = b$.*

(MVP2) *For all $a, b \in B$ and $x \in L(a, b)$, there exists $m \in \max(L(a, b))$ with $m \geq x$.*
$L(a, b) := \{a \wedge f(b);\ f \in G\}$; $\max(L(a, b))$ the set of maximal elements in $L(a, b)$.

MV-pairs–MV-algebras

Theorem 5. (Jenča, 2007) *Let* (B, G) *be an MV-pair. Then* $\sim_G$ *is an effect algebra congruence on* B *and* $B/\sim_G$ *is an MV-effect algebra.*

Theorem 6. (Jenča, 2007) *Let* M *be an MV-effect algebra. Let* $G(M)$ *be the set of all* ψ_M*-preserving automorphisms of* $B(M)$. *Then* $(B(M), G(M))$ *is an MV-pair and the MV-effect algebra* $B(M)/\sim_{G(M)}$ *is isomorphic to* M.

A modification of MV-pairs

Definition 7. *A BG-pair is an* MV**-pair if the following conditions are satisfied:*
(MVP1*) *for any* $a, b \in B$ *and* $f \in G$, *if* $a \perp b$ *and* $a \perp f(b)$ *then there is* $h \in G$ *with* $h(a \vee b) = a \vee f(b)$.

(MVP2*) *Let* $L^+(a,b) = \{f(a) \wedge g(b) : f, g \in G\}$, $\max(L^+(a,b))$ *be the set of maximal elements in* $L^+(a,b)$. *For every* $x \in L^+(a,b)$ *there is an element* $m \in \max(L^+(a,b))$ *such that* $x \leq m$.

MV*pair — MV-algebra

Theorem 8. *Let* (B, G) *be a BG-pair.* (i) *The relation* $\sim_G$ *is an effect algebra congruence iff* (MVP1*) *holds.* (ii) *If* (MVP1*) *holds, then the quotient* $B/\sim_G$ *is an MV-effect algebra iff* (MVP2*) *holds.*

Theorem 9. *Let* (B, G) *be a BG-pair such that (MVP1) is satisfied. Then (MVP2) is satisfied iff* B/G *is an MV-effect algebra.*

Theorem 10. *Let* (B, G) *be an MV-pair, and* I *a G-invariant ideal in* B. *Then* $(B/I, G')$ *is also an MV-pair. Moreover,* $(B/I)/G' \cong (B/G)/(I/G)$.

(Here B/G means $B/\sim_G$, and $g'([a]_I) = [g(a)]_I$).

(MVP1) is stronger that (MVP1*)

Let B be a Boolean algebra with three atoms a_1, a_2, a_3. The mapping f given by

$$f(0) = 0,\ f(a_1) = a_2,\ f(a_2) = a_3,\ f(a_3) = a_1$$

extends to an automorphism of B and $G = \{id, f, f^2\}$ is a subgroup of Aut(B). (MVP1) does not hold: $a_1 \leq a_3'$, $f(a_1) = a_2 \leq a_3'$, but there is no $h \in G$ with $h(a_1) = f(a_1)$ and $h(a_3') = a_3'$. But B/G is the chain $[0] \leq [a] \leq [a'] \leq [1]$.

MV-pairs and states

A *state* on an effect algebra E is a mapping $s : E \longrightarrow [0,1] \subseteq \mathbb{R}$ such that:
(i) $s(1) = 1$; (ii) $s(a + b) = s(a) + s(b)$ whenever $a \perp b$.

States on MV-algebras coincide with states on MV-effect algebras.

$I_s = \{a \in M : s(a) = 0\}$ is an ideal.

Theorem 11. *Let (B, G) be an MV-pair, and let s be a state on B which is G-invariant, that is, $s(f(a)) = s(a)$ for all $f \in G$. Let $M = B/G$, and let $\phi : B \longrightarrow M$ be the canonical morphism. Then $\tilde{s}$ defined by $\tilde{s}(\phi(a)) = s(a)$ is a state on M.*
If s is a state on M, then s_0 defined by $s_0(a) = s(\phi(a))$ is a G-invariant state on B.

0-1 state

If P is a generalized effect algebra (effect algebra without 1), there is an effect algebra $E = P\dot{\cup}(E \setminus P)$ satisfying the diagram (τ is injective, π is surjective effect algebra morphism, and the image of τ is the kernel of π):

$$0 \to P \stackrel{\tau}{\to} E \stackrel{\pi}{\to} \{0, 1\} \to 0,$$

E- unitization of P. An effect algebra E is a unitization of some generalized effect algebra P iff E has a 0-1-state.

Example 12. Let (B,G) be an MV-pair, and let s be a G-invariant 0-1-state on B. Then $\tilde{s}$ is a 0-1-state on $M = B/G$ and M can be written as $M = I_{\tilde{s}} \dot{\cup} I_{\tilde{s}}^{\perp}$, where $I_{\tilde{s}}^{\perp} = \{a' : a \in I_{\tilde{s}}\}$, and M is the unitization of $I_{\tilde{s}}$, which is a generalized MV-effect algebra.

On the other hand, if $M = P\dot{\cup}P^{\perp}$ is a unitization of the generalized MV-effect algebra P, then it has a 0-1-state which extends to a G-invariant 0-1 state for the corresponding MV-pair $(B(M), G(M))$.

Perfect MV-algebra

Example 13. Let M be a perfect MV-algebra, that is, $M = R\dot{\cup}R^{\perp}$, where R is its radical and $R^{\perp} = \{a' : a \in R\}$ its co-radical. Then M is a unitization of R. Since R consists of all infinitesimal elements and 0, there is only one state s on M, namely $s(a) = 0$ whenever $a \in R$, and $s(a) = 1$ if $s \in R^{\perp}$. Accordingly, there is only one G-invariant state s_0 for the MV-pair $(B(M), G(M))$, and s_0 is 0-1. It follows that $B(M) = P\dot{\cup}P^{\perp}$, where $P = \{a \in B(M) : \psi_M(a) \in R\}$.

Theorem 14. *Let M be an MV-algebra, $B(M)$ its R-generated Boolean algebra and $\psi : B(M) \to M$ the corresponding full surjective effect algebra morphism, and let $G(M)$ be the group of all ψ-invariant automorphisms of B.* (i) *For every ψ-invariant state on $B(M)$, $\tilde{s}$ is a state on M, which is a restriction of s to M.* (ii) *For every state on M, $s_0(a) = s(\psi(a))$ is a G-invariant state on $B(M)$. If moreover s is an extremal state on M, then s extends to a boolean algebra homomorphism s^* from $B(M)$ onto $B(s(M))$, and $s \circ \psi(a) = \psi_1 \circ s^*(a)$.*

Commuting diagram

$$\begin{array}{ccc} M & \xleftarrow{\quad\psi\quad} & B(M) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s*} \\ s(M) & \xleftarrow{\quad\psi_1\quad} & B(s(M)) \end{array}$$

Remark 15. (B, G) – an MV-pair, s – a G-invariant state on B. I_s is a G-invariant ideal of B.
$[a]_s$ the equivalence class in B/I_s containing $a \in B$.
Then $s^0([a]_s) := s(a)$ is a state on B/I_s.
$(B/I_s, G')$ is an MV-pair, so
$\widetilde{s^0}([[a]_s]_{G'}) = s^0([a]_s) = s(a)$ is a state on $(B/I_s)/G'$.
On the other hand, since s is G-invariant, we may defined the state $\tilde{s}([a]_G) := s(a)$ on B/G.
And since I_s/G is an ideal of B/G, we may define the state $\tilde{s}_0([[a]_G]_s) = \tilde{s}([a]_G) = s(a)$ on $(B/G)/(I_s/G)$.
Clearly, $\widetilde{s^0} = \tilde{s}_0$.

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