# Rings of endomorphisms of Abelian groups with special 0-neighborhoods

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## Abstract

Properties of a 0-neighborhood of a topological ring have an influence to the global properties of it. For instance, commutativity of a connected ring is equivalent to commutativity of a neighborhood of 0. The connections between local and global properties of a topological ring are weaker in the case of disconnected rings.
We study the influence of properties of a neighborhood of 0 of a topological ring $(\mathrm{End}(A), \mathcal{T})$,where $\mathrm{End}(A)$ is the ring of endomorphisms of an Abelian group $A$ and $\mathcal{T}$ the finite topology. We show that although the rings of endomorphisms of Abelian groups are totally disconnected, there is a close relation between local and global properties.

# Endomorphism rings with special 0-neighborhoods

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4. $J[\mathrm{End}(A)]$ *is open.*

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*If $A$ is an Abelian separable reduced $p$-group and $\mathrm{End}(A)$ have a 0-neighborhood without zero-divisors, then $A$ is finite.*

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*If $A$ is a torsion Abelian group and $\mathrm{End}(A)$ have a nilpotent 0-neighborhood, then $A$ is a finite direct sum of p-groups.*

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**Theorem**

*If* $\mathrm{End}({}_{\mathbb{Z}_p}M)$ *is compact then* $M$ *is a* $p$*-group.*

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Let $M_R$ be a discrete right $R$-module.
Two elements $m, m'$ have the same order if the modules $mR$ and $m'R$ are isomorphic.
Johnson and Wong have called a module $M_R$ *quasi-injective* if every partial endomorphism of $M_R$ can be extended to a full endomorphism. We have the following theorem:

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**Theorem**

*Let $M_R$ be a quasi-injective module. Then $\mathrm{End}(M_R)$ is compact $\Leftrightarrow$ for every $m \in M_R$ the set of $m'$ having the same order as $m$ is finite.*