Dualities and colourings

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Summer School on Algebra and Ordered Sets
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Třešť

Natural Dualities

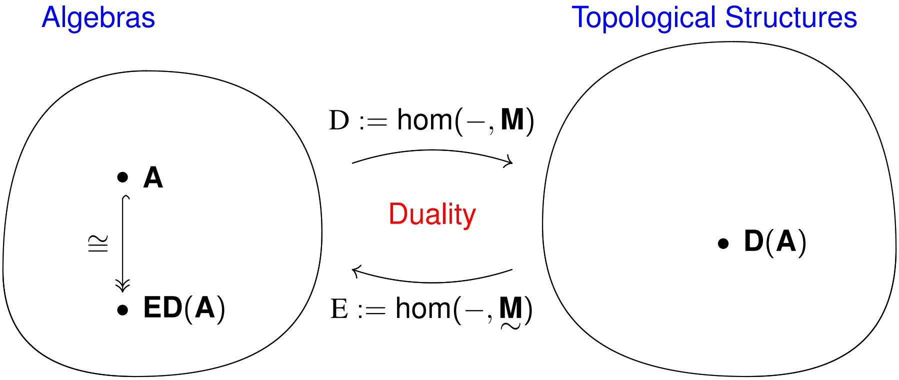
General Duality Theory (1980)

Algebras

Topological Structures

Natural Dualities

General Duality Theory (1980)



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$\mathbf{M} = \langle M; F \rangle$

(a finite algebra)

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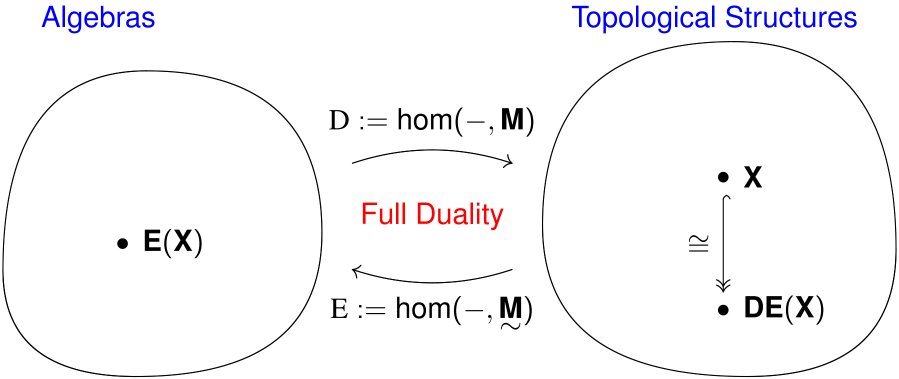

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Finite-level Dualities

A finite-level duality (full duality, strong duality) means that the corresponding concepts are defined between the categories $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$ of finite algebras and structures.

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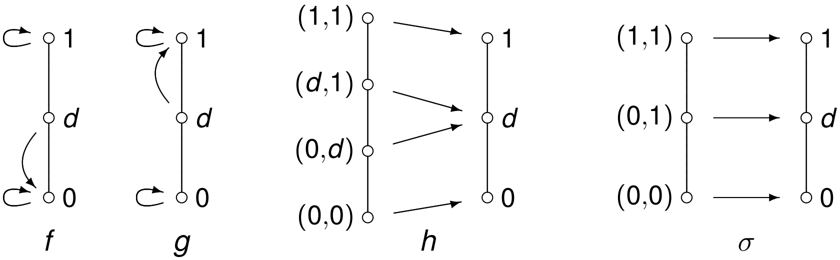

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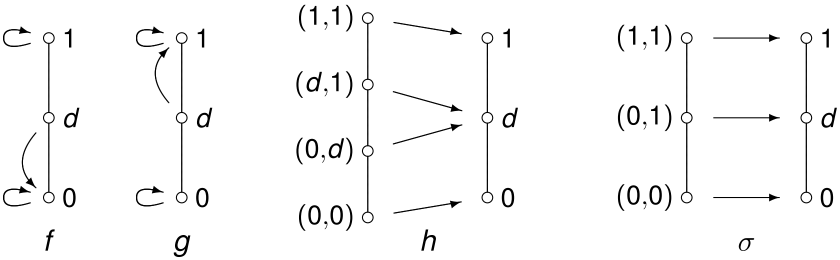

- Let $\underset{\sim}{\mathbf{3}} := \langle \{0, d, 1\}; f, g \rangle$, $\underset{\sim}{\mathbf{3}}_h := \langle \{0, d, 1\}; f, g, h \rangle$,
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- NO, in general: a duality constructed by Clark, Davey, Willard [June 2006] (Algebra Universalis 57 (2007), 375-381).

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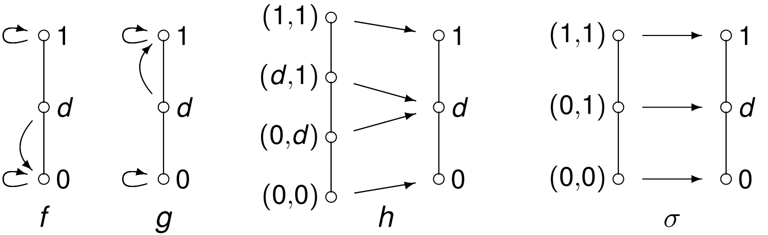

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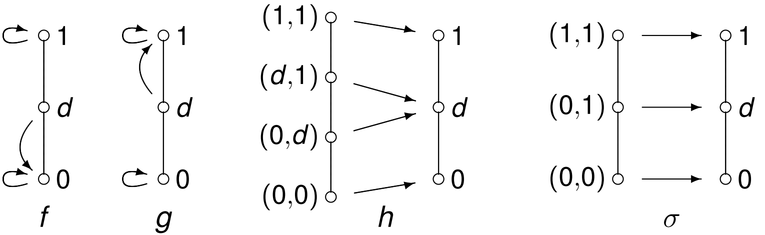

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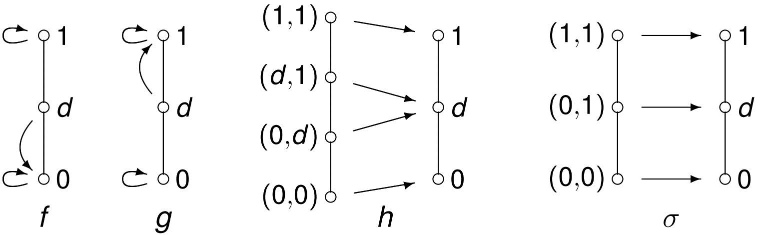

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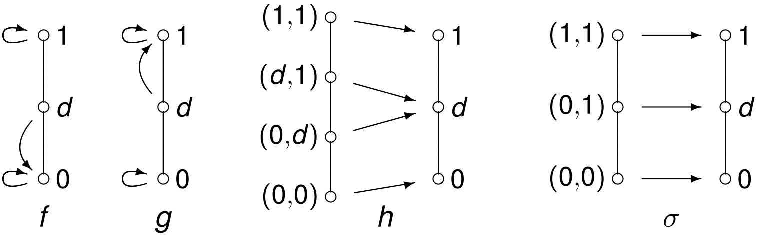

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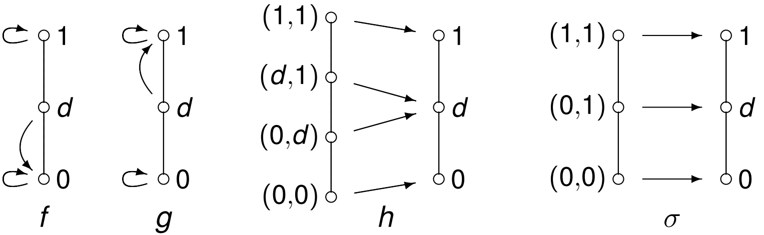

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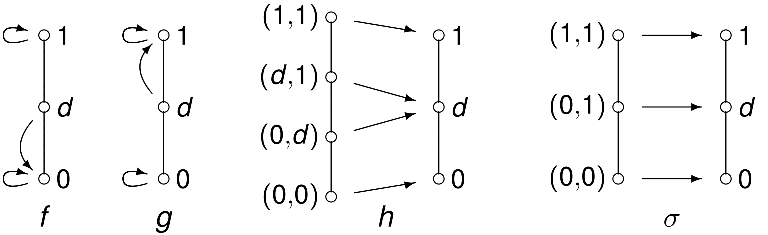

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- **but not conversely**:
  - $\{0, 1\}$ is closed under f, g and h, but not under σ, so σ cannot be defined in terms of f, g and h.

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What lies between $\underset{\sim}{\mathbf{3}}_h$ and $\underset{\sim}{\mathbf{3}}_\sigma$?

Encoding algebraic relations as coloured ordered sets

- We can use Priestley duality to encode the algebraic relation $L = \{0, d, 1\}^2 \setminus \{(1, 0), (0, 1)\}$ on $\underline{\mathbf{3}}$ as a *coloured ordered set* **C**.

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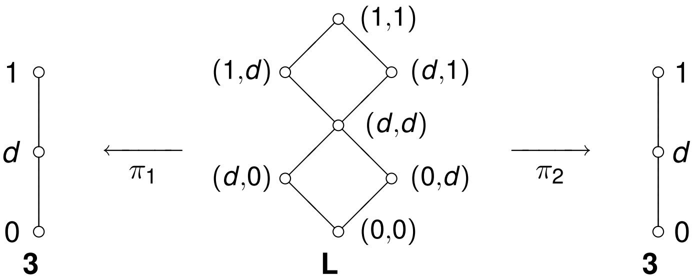

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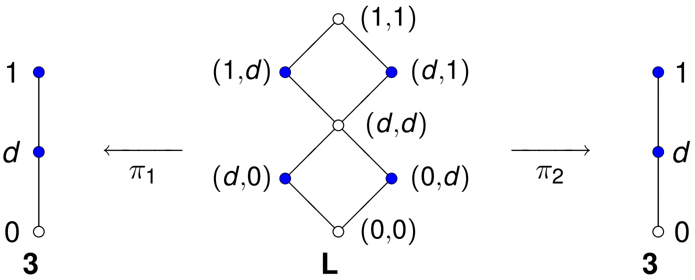

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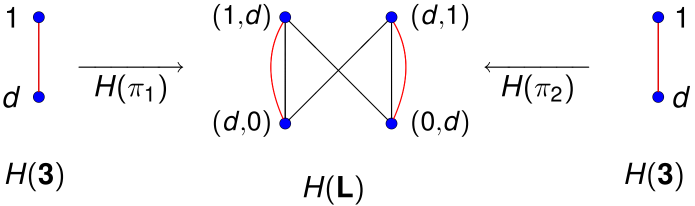

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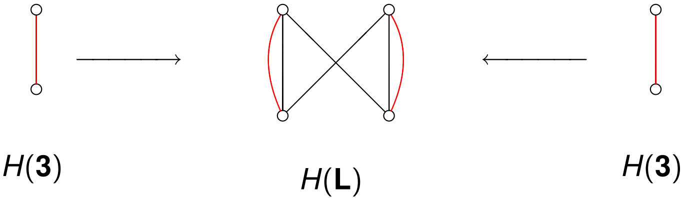

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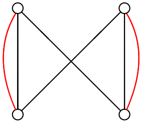

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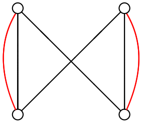

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- The red edges remember the coordinate projections up to a permutation.

Recovering algebraic relations from coloured posets

- We can recover the algebraic relation L back from the coloured ordered set **C**.

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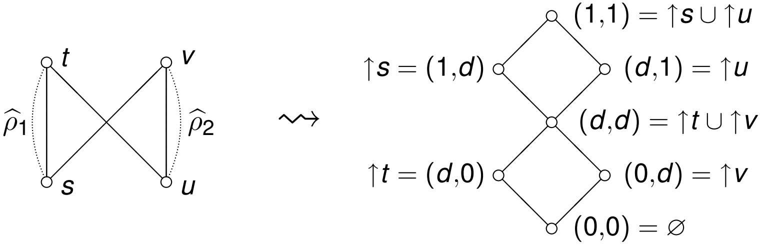

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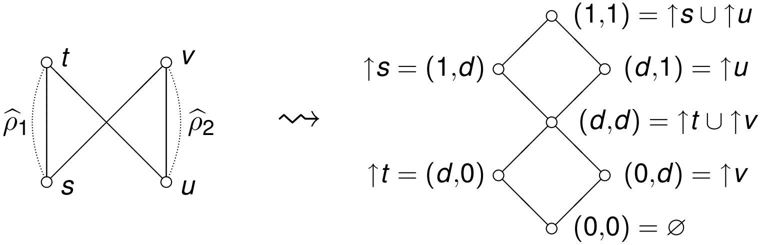

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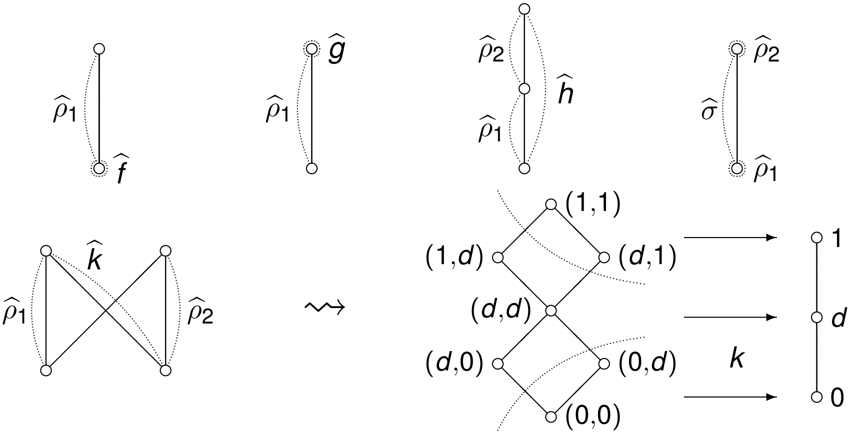

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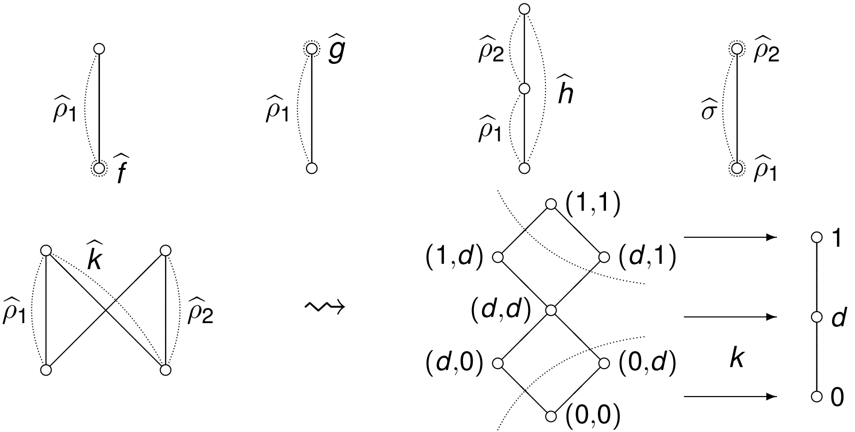

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A homomorphism between coloured ordered sets must preserve both $\leqslant$ and $\lhd$.

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A quasi-order on coloured ordered sets

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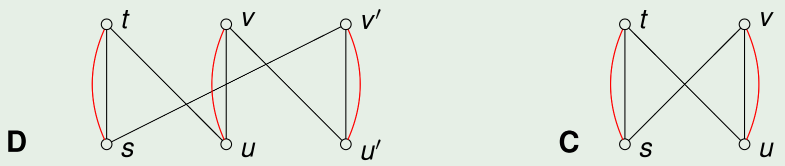

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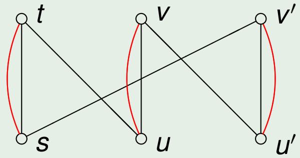
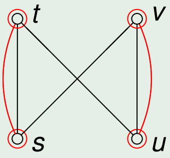


$\mathbf{C}_0$



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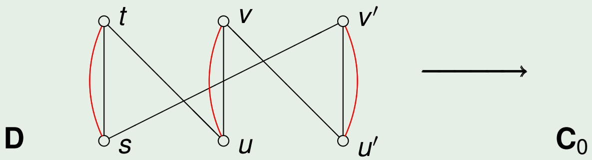

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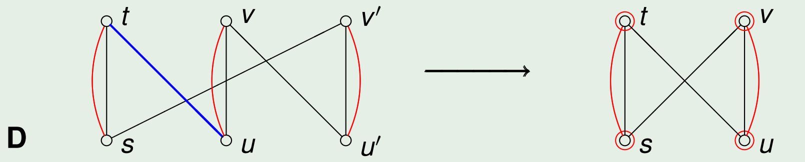

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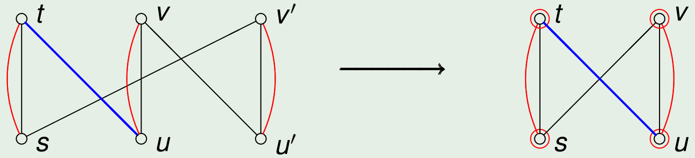

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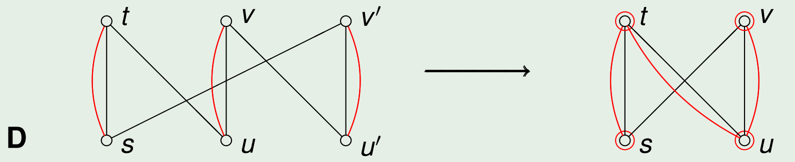

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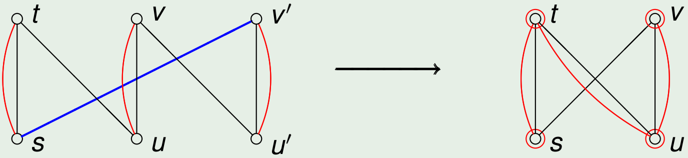

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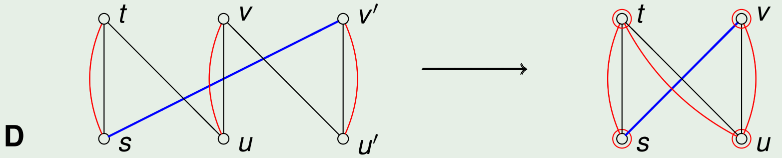

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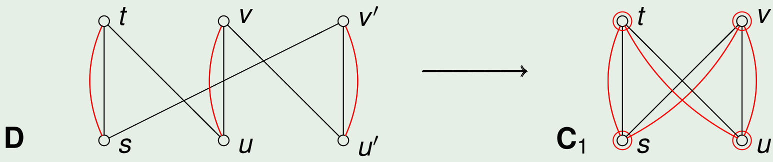

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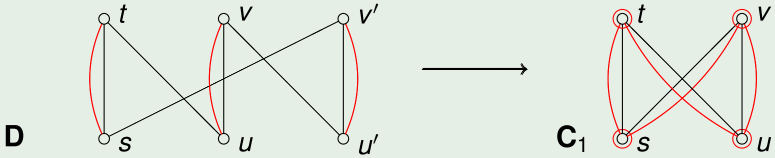
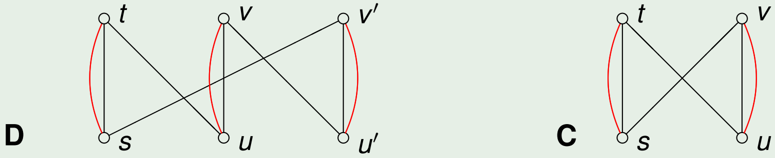


D cannot be coloured using C



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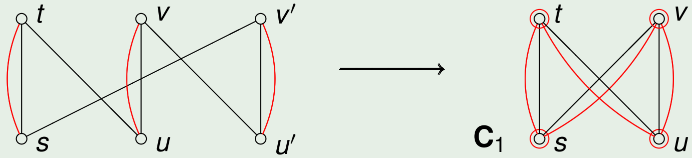

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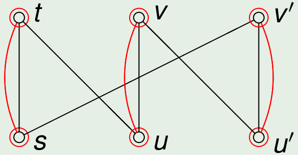

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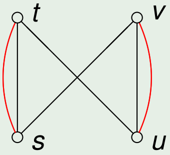

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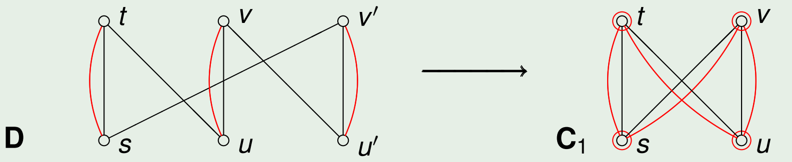

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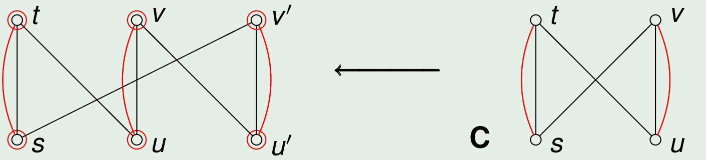

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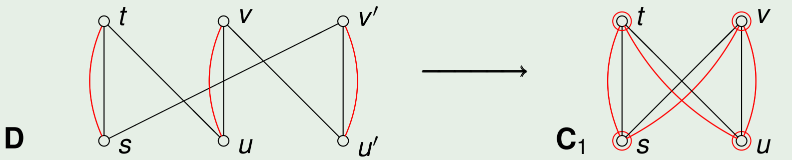

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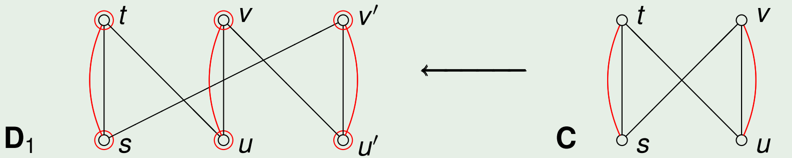

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Theorem

The ordered set $\mathcal{C}$ is isomorphic to the lattice $\mathcal{F}(\underline{\mathbf{3}})$ of full dualities for $\mathcal{D}_{\text{fin}}$ based on $\underline{\mathbf{3}}$.

Illustrations 1

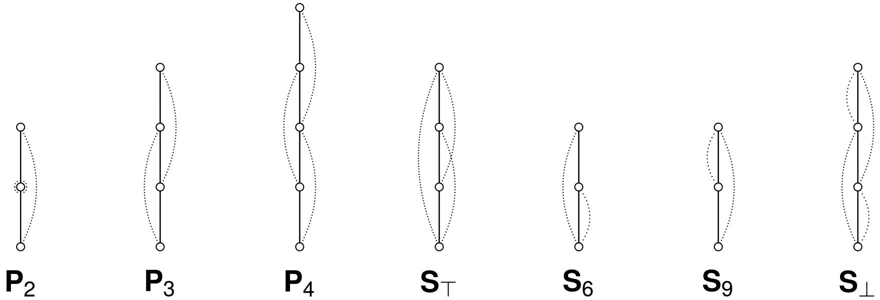

Figure: Some different coloured ordered sets

Illustrations 2

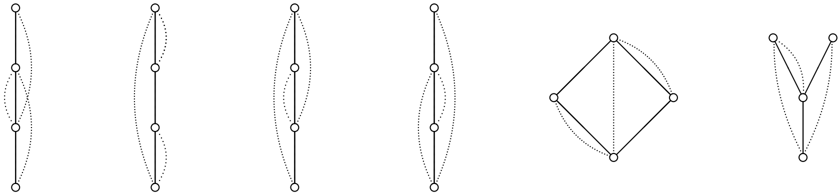

Figure: Coloured ordered sets equivalent to $\mathbf{S}_{\top}$

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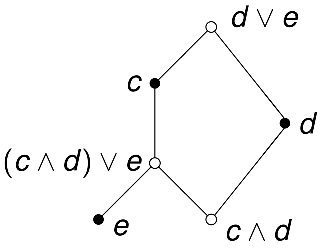
- Meet-sausage Problem: Is $\mathbf{S}_\bot \equiv \mathbf{S}_6 \wedge \mathbf{S}_9$?
- The following easy lemma allows us to show that $\mathfrak{C}$, and therefore $\mathcal{F}(\underline{\mathbf{3}})$, is non-modular without actually calculating a meet.

The lattice is non-modular

Lemma

Let $\mathcal{L}$ be a lattice and let $c, d, e \in L$. Assume that c is join-irreducible in $\mathcal{L}$, and that $e < c \leqslant d \vee e$ and $c \nleqslant d$. Then the lattice $\mathcal{L}$ is not modular. Indeed, the pentagon $\mathcal{N}_5$ embeds into $\mathcal{L}$ as shown below.



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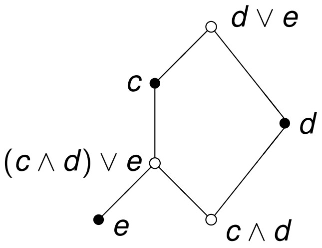
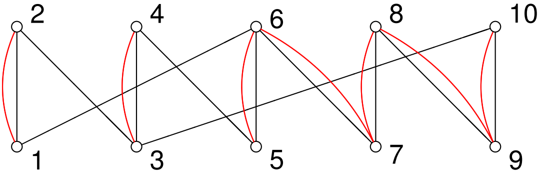


To prove that $\mathcal{C}$ is non-modular it suffices to find three coloured ordered sets **C**, **D** and **E** satisfying the conditions of this lemma.

The coloured ordered sets **C**, **D** and **E**

C



C is join-irreducible,
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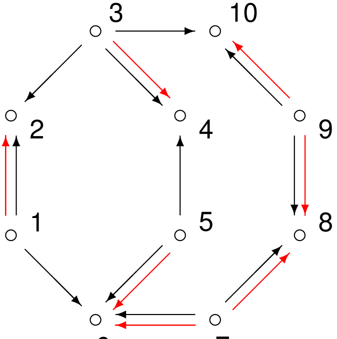
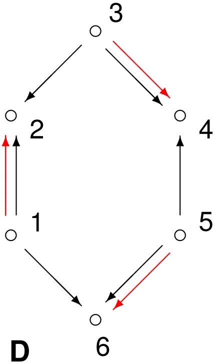
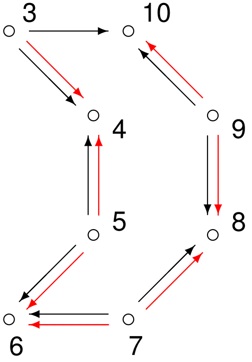


C



D



E

The lattice $\mathcal{F}(\underline{\mathbf{3}})$ is as big as possible

Theorem

- The lattice $\mathcal{F}(\underline{\mathbf{3}})$ has cardinality $2^{\aleph_0}$.
- The lattice $\mathcal{F}(\underline{\mathbf{3}})$ contains a countably infinite antichain.
- The lattice $\mathcal{F}(\underline{\mathbf{3}})$ contains an uncountable chain.

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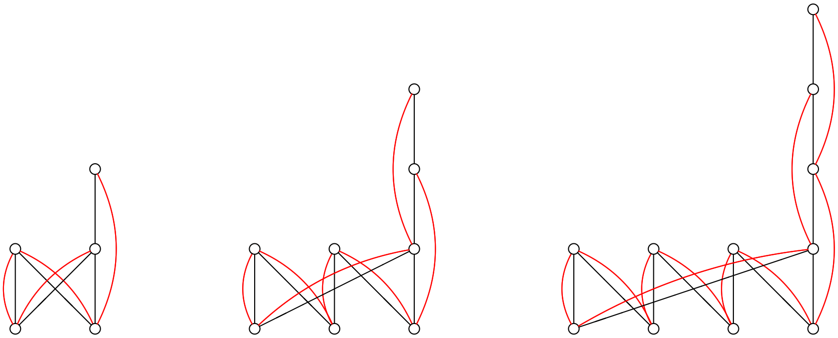
Proof

Embed the ordered set $\langle \wp(\mathbb{N}); \subseteq \rangle$ into $\mathfrak{C}$ via the coloured ordered sets $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, . . . which form an independent antichain in $\mathfrak{C}$.



$\mathbf{W}_1$ $\mathbf{W}_2$ $\mathbf{W}_3$

An infinite descending chain in $\mathcal{F}(\underline{\mathbf{3}})$

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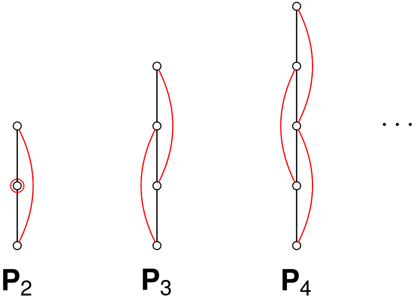
Proof

Show that the coloured ordered sets $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \ldots$ form an infinite descending chain $\mathbf{P}_2 > \mathbf{P}_3 > \mathbf{P}_4 > \cdots$ in $\mathcal{C}$.



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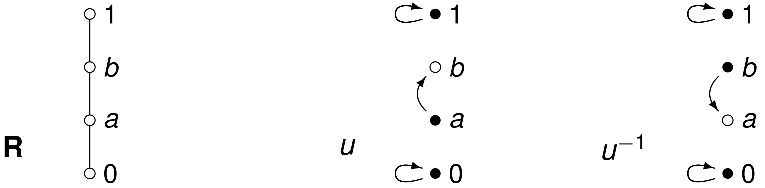
- The three-element bounded lattice $\underline{\mathbf{3}}$ is the first example where the lattice $\mathcal{F}(\mathbf{M})$ has been proved to be infinite (**Davey, Haviar and Pitkethly [2006-8]**).

The Negative Solution of the Full vs Strong Problem:
The Algebra and the Alter Ego

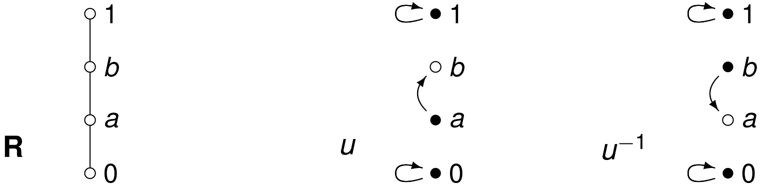


Full Does Not Imply Strong! [Clark, Davey, Willard (2006)]

Let $\mathbf{R} := \langle \{0, a, b, 1\}; t, \vee, \wedge, 0, 1 \rangle$, where $0 < a < b < 1$ and the operation t is the ternary discriminator.

The Negative Solution of the Full vs Strong Problem:

The Algebra and the Alter Ego



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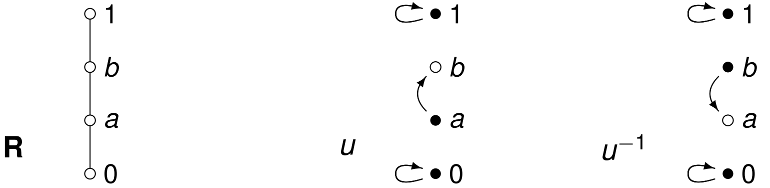

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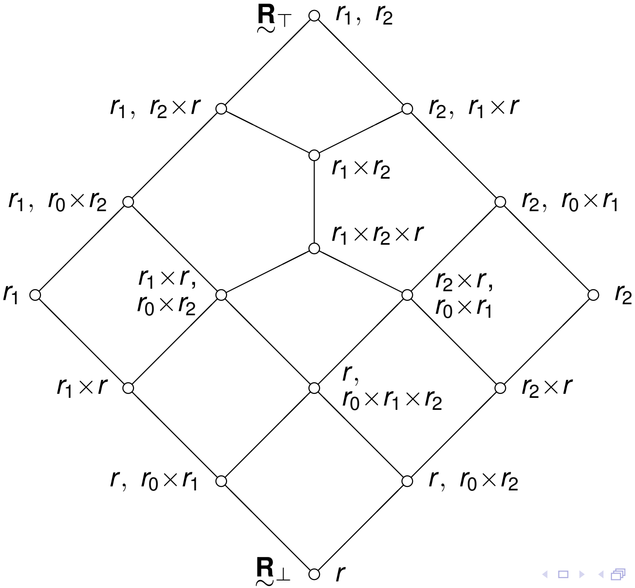
- $\underset{\sim}{\mathbf{R}}_{\top} := \langle \{0, a, b, 1\}; u, u^{-1}, \mathcal{T} \rangle$ yields a strong duality on $\mathbb{ISP}(\mathbf{R})$.
- $\underset{\sim}{\mathbf{R}}_{\perp} := \langle \{0, a, b, 1\}; \mathrm{graph}(u), \mathcal{T} \rangle$ yields a full but not strong duality on $\mathbb{ISP}(\mathbf{R})$.

The Negative Solution:
The Lattice of All Full Dualities on $\mathbb{ISP}(\mathbf{R})$ [Davey, Pitkethly, Willard (2007)]



$$r = \mathrm{graph}(u)$$
$$r_0 = \mathrm{fix}(u)$$
$$r_1 = \mathrm{dom}(u)$$
$$r_2 = \mathrm{ran}(u)$$