Congruence lifting of semilattice diagrams

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August 28, 2008

Introduction

Problem. For a given class $\mathcal{K}$ of algebras describe
$\mathrm{Con}\ \mathcal{K} =$ all lattices isomorphic to $\mathrm{Con}\, A$ for some $A \in \mathcal{K}$.

Or, at least
for given classes $\mathcal{K}$, $\mathcal{L}$ determine if $\mathrm{Con}\ \mathcal{K} = \mathrm{Con}\ \mathcal{L}$
$(\mathrm{Con}\ \mathcal{K} = \mathrm{Con}\ \mathcal{L})$.

Especially, for finitely generated varieties $\mathcal{K}$, $\mathcal{L}$ we have an algorithmic problem.

Con functor

The Con functor:

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}\, f: \mathrm{Con}\, A \to \mathrm{Con}\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

Lifting of semilattice morphisms

Let

- $\varphi : \; S \to T$ be a $(\vee, 0)$-homomorphisms of lattices;
- $f : \; A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1 : \; S \to \operatorname{Con} A$, $\psi_2 : \; T \to \operatorname{Con} B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ {\scriptstyle \psi_1}\big\downarrow & & {\scriptstyle \psi_2}\big\downarrow \\ \operatorname{Con} A & \xrightarrow{\operatorname{Con} f} & \operatorname{Con} B \end{array}$$

commutes.

Diagrams indexed by posets 1

Let

- $(P, \leq)$ be a poset;
- $\mathcal{K}$ be a category of algebras

Definition. A $(P, \leq)$-indexed diagram in $\mathcal{K}$ is a functor

$$\mathcal{A} : \ (P, \leq) \to \mathcal{K}.$$

Diagrams indexed by posets 2

That means:

- an algebra $\mathcal{A}(j) \in \mathcal{K}$ for every $j \in P$;
- a homomorphisms $\mathcal{A}(j,k) : \ \mathcal{A}(j) \to \mathcal{A}(k)$ for every $j \leq k$;

such that

- $\mathcal{A}(j,j) = \mathrm{id}(\mathcal{A}(j))$ for every $j \in P$;
- $\mathcal{A}(j,k) \circ \mathcal{A}(i,j) = \mathcal{A}(i,k)$ for every $i \leq j \leq k$.

Lifting of diagrams

Let P be a poset and let

- $\mathcal{D}: \; P \to \mathcal{S}$ be a diagram of $(\vee, 0)$-semilattices;
- $\mathcal{A}: \; P \to \mathcal{K}$ be a diagram of algebras;

We say that $\mathcal{A}$ *lifts* $\mathcal{D}$, if there are isomorphisms $\psi_j: \; \mathcal{D}(j) \to \operatorname{Con} \mathcal{A}(j)$ such that

$$\begin{array}{ccc} \mathcal{D}(j) & \xrightarrow{\mathcal{D}(j,k)} & \mathcal{D}(k) \\ \psi_j \downarrow & & \psi_k \downarrow \\ \operatorname{Con} \mathcal{A}(j) & \xrightarrow{\operatorname{Con} \mathcal{A}(j,k)} & \operatorname{Con} \mathcal{A}(k) \end{array}$$

commutes for every $j \leq k$.

Results of P. Gillibert 1

Let $\mathcal{K}$, $\mathcal{L}$ be finitely generated congruence distributive varieties. Put

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

(or ∞).

Theorem

TFAE

- $\mathrm{Con}\,\mathcal{K} \nsubseteq \mathrm{Con}\,\mathcal{L}$;
- *there exists a diagram of finite $(\vee, 0)$-semilattices indexed by $\{0,1\}^n$ (for some n) liftable in $\mathcal{K}$ but not in $\mathcal{L}$*

Results of P.Gillibert 2

Theorem

(2) implies (1), where

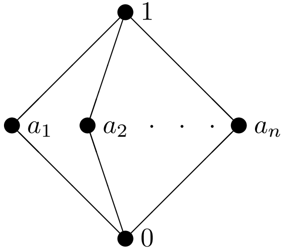
- $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_n$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of* $n+1$ *finite chains liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

If $n = 0$ *then also (1)*$\Longrightarrow$ *(2).*

Question. What about (1)$\Longrightarrow$(2) for $n > 0$?

Critical point aleph2

Let $\mathcal{M}_n^{01}$ be the variety of bounded lattices generated by



Critical point aleph2

Theorem

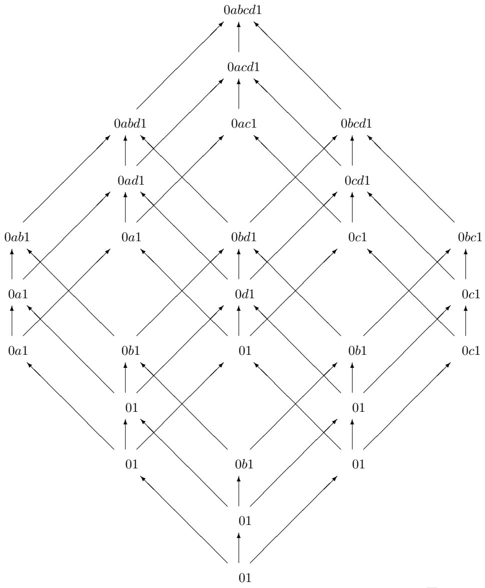
(MP 1998, 2000)

$$\mathrm{Crit}(\mathcal{M}_{n+1}^{01}, \mathcal{M}_n^{01}) = \aleph_2$$

for every $n \geq 3$.

Question. Is there a diagram indexed by a product of 3 finite chains liftable in $\mathcal{M}_{n+1}^{01}$ but not in $\mathcal{M}_n^{01}$?

M3 versus M4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General construction 1

Consider the following three linear orders on the set $\{1, 2, \ldots, n\}$:

$$1 <_1 2 <_1 3 <_1 \cdots <_1 n;$$

$$1 <_2 n <_2 n-1 <_2 n-2 <_2 \cdots <_2 2;$$

$$2 <_3 n <_3 n-1 <_3 \cdots <_3 3 <_3 1.$$

Let Z_k^i be the unique k-element lower subset of the ordered set $(\{1, \ldots, n\}, \leq_i)$ ($i \in \{1, 2, 3\}$, $1 \leq k \leq n$) and

$$Z(j, k, l) = Z_{j+2}^1 \cap Z_{k+2}^2 \cap Z_{l+2}^3.$$

General Construction 2

Define a diagram $\mathcal{A}: \{0,1,\dots,n-2\}^3 \to \mathcal{M}_n^{01}$ by

- $\mathcal{A}(j,k,l)$ is a free algebra in $\mathcal{M}_n^{01}$ generated by $Z(j,k,l)$;
- all $\mathcal{A}$-morphisms are set inclusions.

Theorem

For any $n > 3$, $\operatorname{Con} \circ \mathcal{A}$ is not liftable in $\mathcal{M}_{n-1}$.