Representations of algebraic distributive lattices

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S-valued distance

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An **S**-valued distance is a map $\delta\colon X \times X \to \mathbf{S}$, where X is a set and **S** is a $(\vee, 0)$-semilattice satisfying:

- $\delta(x, x) = 0$;
- $\delta(x, y) = \delta(y, x)$;
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The **S**-valued distance δ satisfies the *V-condition* if whenever $\delta(x, y) \leq \alpha \vee \beta$ there is a chain $x = z_0, z_1, \dots, z_n = y$ in X such that $\delta(z_i, z_{i+1}) \leq \alpha$ for i even and $\delta(z_i, z_{i+1}) \leq \beta$ for i odd.

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If the length of such a chain can be always bounded by n, we say that δ is a V-ditance of type n and $1/2$.

m-permutable congruences

Proposition

Let $\mathbf{A}$ *be an algebra with* $(m+1)$*-permutable congruences. Then the* $(\vee, 0)$*-semilattice* $\mathrm{Con}_c(\mathbf{A})$ *(of all finitely generated (= compact) congruences) is join generated by the range of a* V*-distance of type* n*.*

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Example

– Let **M** be a module over a ring **R**. Then the $(\vee, 0)$-semilattice $\mathrm{SubF}(\mathbf{M})$ of finitely generated submodules of **M** is join-generated by the range of the V-distance δ of type 1 defined by $\delta(x, y) = (x - y)\mathbf{R}$.

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- If $\mathbf{G}$ is a group, then the $(\vee, 0)$-semilattice $\mathrm{NSubF}(\mathbf{G})$ of finitely generated normal subgroups of $\mathbf{G}$ is join-generated by the range of the V-distance δ of type 1 defined by $\delta(x, y) = \langle xy^{-1} \rangle^{\mathbf{G}}$.

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We say that a $(\vee, 0)$-semilattice $\mathbf{S}$ satisfies $\mathrm{WURP}^{=}(e)$, for some $e \in \mathbf{S}$, if there is a positive integer m such that for all families $(a_i \mid i \in I)$, $(b_i \mid i \in I)$ of elements of $\mathbf{S}$ with $e \leq a_i \vee b_i$ for all I, there are subsets $I_0, \ldots, I_{m-1}$ of I such that $I = \bigcup_{i<m} I_i$ and elements $c_{i,j}$, $i, j \in I$ of $\mathbf{S}$ satisfying

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Application of $\mathrm{WURP}^{=}$

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Let $\mathcal{V}$ *be a non-distributive variety of lattices and let* $\mathbf{F}$ *be a free (bounded) lattice in* $\mathcal{V}$ *generated by at least* $\aleph_2$ *elements. Then* $\mathrm{Con}_c(\mathbf{F})$ *does not satisfy* $\mathrm{WURP}^{=}$*.*

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Thus there is no V*-ditance* δ *of type* $3/2$ *with range join-generating* $\mathrm{Con}_c \mathbf{F}$ *and there is no algebra* $\mathbf{A}$ *with almost permutable congruences (e.g., an* $\mathbf{R}$*-module or a group) such that* $\mathrm{Con} \mathbf{F} \simeq \mathrm{Con} \mathbf{A}$*.*

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Let k be a positive integer, let Ω be a set. For a map $\Psi\colon [\Omega]^{k-1} \to [\Omega]^{<\omega}$, we say that a k-element subset B of Ω is free (with respect to Ψ) if $b \notin \Psi(B \smallsetminus \{b\})$, for all $b \in B$.

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The general counter-example was found by F. Wehrung as a colision of the Evaporation lemma (which in some sense replace uniform refinement properties) and the Erosion lemma. The colision is gained by applying the Kuratowski free set theorem. However to obtain a counter-example of (optimal) cardinality $\aleph_2$, we need to use free trees instead of free sets.

Free trees

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An indexed subset $\mathcal{T} = \{\alpha_f \mid f\colon n \to k\}$ of Ω is called a free k-tree of height n (with respect to Φ) if the sets

$$\Phi\Big(\underbrace{\{\alpha_f \mid f\colon n \to k \mid f \restriction [m+1, n-1] = g \text{ and } f(m) \neq i\}}_{\mathcal{T}(g,\neg i)}\Big),$$

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are disjoint for all $m < n$, all maps $g\colon [m+1, n-1] \to k$, and all $i < k$.

Lemma

Let Ω be a set, let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ be a map, and let $0 < k$ and n be natural numbers. Then every subset X of Ω of cardinality at least $\aleph_{k-1}$ contains a free k-tree of height n.

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