A Cayley Theorem for distributive lattices and for algebras with binary and nullary operations

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Cayley's Theorem provides a well-known representation of groups by means of certain unary functions (the so-called permutations) with composition as its binary operation. For Boolean algebras, a representation via binary functions (the so-called guard functions) was settled by Bloom, Ésik and Manes. A similar approach was used by the first author for the so-called q-algebras. Here we will firstly present a representation of distributive lattices by means of binary functions.

First we define an algebra in which we will embed the distributive lattices.

Definition 1. *For every set* M *let* $\mathcal{F}(M)$ *denote the algebra* $(M^{M^2}, \diamond, *)$ *of type* $(2,2)$ *defined by* $(f \diamond g)(x,y) := f(g(x,y),y)$ *and* $(f * g)(x,y) := f(x, g(x,y))$ *for all* $f, g \in M^{M^2}$ *and* $x, y \in M$.

$\mathcal{F}(M)$ is not a lattice, but the operations $\diamond$ and $*$ are associative.

Definition 2. *For every lattice* $(L, \vee, \wedge)$ *and every* $a \in L$ *let* f_a *denote the mapping* $(x, y) \mapsto (a \vee x) \wedge y$ *from* L^2 *to* L *and* φ *the mapping* $a \mapsto f_a$ *from* L *to* L^{L^2}*.*

Now we can state our first result:

Theorem 1. *For every distributive lattice* $\mathcal{L} = (L, \vee, \wedge)$ *the mapping* φ *is an embedding of* $\mathcal{L}$ *into* $\mathcal{F}(L)$*.*

That for a distributive lattice $(L, \vee, \wedge)$ the algebra $(\varphi(L), \diamond, *)$ is a lattice follows from a more general result. In order to be able to formulate this result in a concise way we make the following definition:

Definition 3. *We call a* subuniverse A *of* $\mathcal{F}(M)$ full *if for all* $f, g \in A$ *and* $x, y, z, u \in M$

(i) $f(x, x) = x$,

(ii) $f(g(x, y), g(z, u)) = g(f(x, z), f(y, u))$,

(iii) $f(f(x, g(x, y)), y) = f(x, f(g(x, y), y)) = f(x, y)$.

Now we can prove

Theorem 2. *If A is a full subuniverse of $\mathcal{F}(L)$ then $(A, \diamond, *)$ is a lattice.*

That for a distributive lattice $(L, \vee, \wedge)$ the algebra $(\varphi(L), \diamond, *)$ is a lattice now follows from

Theorem 3. *If* $\mathcal{L} = (L, \vee, \wedge)$ *is a distributive lattice then* $\varphi(L)$ *is a full subuniverse of* $\mathcal{F}(L)$ *and hence* $(\varphi(L), \diamond, *)$ *is a lattice isomorphic to* $\mathcal{L}$*.*

The Cayley Theorem for monoids (which is essentially the same as that for groups) is well known and a Cayley Theorem for distributive lattices was presented. We will present a common generalization of both theorems.

In the following let n be an arbitrary, but fixed positive integer.

Definition 4. *Let $\mathcal{V}_n$ denote the variety of all algebras $(A, \bullet_1, \ldots, \bullet_n)$ of type $(2, \ldots, 2)$ satisfying the identities*

$$(\ldots((x \bullet_i y) \bullet_1 x_1) \bullet_2 \ldots) \bullet_n x_n =$$
$$= (\ldots((((\ldots(x \bullet_1 x_1) \bullet_2 \ldots) \bullet_{i-1} x_{i-1}) \bullet_i$$
$$\bullet_i((\ldots(y \bullet_1 x_1) \bullet_2 \ldots) \bullet_n x_n)) \bullet_{i+1}$$
$$\bullet_{i+1} x_{i+1}) \bullet_{i+2} \ldots) \bullet_n x_n$$

for $i = 1, \ldots, n$.

Example 1. $\mathcal{V}_1$ is the variety of semigroups.

Example 2. Since an algebra $(A, \vee, \wedge)$ of type $(2, 2)$ belongs to $\mathcal{V}_2$ if it satisfies the identities

$$((x \vee y) \vee z) \wedge u = (x \vee ((y \vee z) \wedge u)) \wedge u$$

$$((x \wedge y) \vee z) \wedge u = (x \vee z) \wedge ((y \vee z) \wedge u),$$

$\mathcal{V}_2$ includes the variety of distributive lattices because for arbitrary elements x, y, z, u of a distributive lattice $(A, \vee, \wedge)$ it holds

$$\begin{aligned} &((x \vee y) \vee z) \wedge u = \\ &= (x \wedge u) \vee (y \wedge u) \vee (z \wedge u) = \\ &= (x \wedge u) \vee ((y \vee z) \wedge u) = \\ &= (x \vee ((y \vee z) \wedge u)) \wedge u \end{aligned}$$

and

$$((x \wedge y) \vee z) \wedge u = (x \vee z) \wedge (y \vee z) \wedge u =$$

$$= (x \vee z) \wedge ((y \vee z) \wedge u).$$

More generally, $\mathcal{V}_2$ includes the variety so-called **solid semirings**. These semirings are defined as algebras of type $(2, 2)$ having the property that both operations are associative and distributive with respect to each other.

Next we want to map our algebras homomorphically into certain algebras of functions. For this purpose we define

Definition 5. *For all algebras* $(A, \bullet_1, \ldots, \bullet_n)$ *of type* $(2, \ldots, 2)$ *and all* $a \in A$ *let* f_a *denote the mapping from* A^n *to* A *defined by*

$$f_a(x_1, \ldots, x_n) := (\ldots(a \bullet_1 x_1) \bullet_2 \ldots) \bullet_n x_n$$

for all $x_1, \ldots, x_n \in A$. *For every set* A *and every* $i \in \{1, \ldots, n\}$ *let* $\circ_i$ *denote the binary operation on* A^{A^n} *defined by the following composition of mappings*

$$(f \circ_i g)(x_1, \ldots, x_n) :=$$

$$f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n)$$

for all $f, g \in A^{A^n}$ *and all* $x_1, \ldots, x_n \in A$.

Now we can state

Theorem 4. *If* $\mathcal{A} = (A, \bullet_1, \ldots, \bullet_n) \in \mathcal{V}_n$ *then* $a \mapsto f_a$ *is a homomorphism from* $\mathcal{A}$ *to* $(A^{A^n}, \circ_1, \ldots, \circ_n)$.

Remark. It was shown that for distributive lattices $(A, \vee, \wedge)$ the homomorphism of Theorem 4 is in fact injective and hence an embedding. Since $a, b \in A$ and $f_a = f_b$ together imply

$$\begin{aligned} a &= (a \vee b) \wedge a = f_a(b, a) = f_b(b, a) = \\ &= (b \vee b) \wedge a = b \wedge a = a \wedge b = (a \vee a) \wedge b = \\ &= f_a(a, b) = f_b(a, b) = (b \vee a) \wedge b = b. \end{aligned}$$

Hence we obtain the Cayley Theorem for distributive lattices already presented.

Definition 6. *Let $\mathcal{V}_{n0}$ denote the variety of all algebras $(A, \bullet_1, \ldots, \bullet_n, e_1, \ldots, e_n)$ of type $(2, \ldots, 2, 0, \ldots, 0)$ satisfying the identities*

$$(\ldots((x \bullet_i y) \bullet_1 x_1) \bullet_2 \ldots) \bullet_n x_n =$$
$$(\ldots((((\ldots(x \bullet_1 x_1) \bullet_2 \ldots) \bullet_{i-1} x_{i-1}) \bullet_i$$
$$\bullet_i((\ldots(y \bullet_1 x_1) \bullet_2 \ldots) \bullet_n x_n)) \bullet_{i+1}$$
$$\bullet_{i+1} x_{i+1}) \bullet_{i+2} \ldots) \bullet_n x_n$$

for $i = 1, \ldots, n$ and the identity

$$(\ldots(x \bullet_1 e_1) \bullet_2 \ldots) \bullet_n e_n = x.$$

Example 3. $\mathcal{V}_{10}$ is the variety of semigroups having a right unit and hence V_{10} includes the variety of monoids.

Example 4. $\mathcal{V}_{20}$ consists of all algebras $(A, \vee, \wedge, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the identities

$$((x \vee y) \vee z) \wedge u = (x \vee ((y \vee z) \wedge u)) \wedge u$$
$$((x \wedge y) \vee z) \wedge u = (x \vee z) \wedge ((y \vee z) \wedge u)$$
$$(x \vee 0) \wedge 1 = x.$$

Since $\mathcal{V}_2$ includes the variety of distributive lattices and for arbitrary elements x of a bounded distributive lattice $(A, \vee, \wedge, 0, 1)$ it holds $(x \vee 0) \wedge 1 = x$, $\mathcal{V}_{20}$ includes the variety of bounded distributive lattices considered as algebras of the form $(A, \vee, \wedge, 0, 1)$.

Now we can state and prove the general Cayley Theorem.

Theorem 5. *If* $\mathcal{A} = (A, \bullet_1, \ldots, \bullet_n, e_1, \ldots, e_n) \in \mathcal{V}_{n0}$ *then* $a \mapsto f_a$ *is an embedding of* $\mathcal{A}$ *into* $(A^{A^n}, \circ_1, \ldots, \circ_n, f_{e_1}, \ldots, f_{e_n})$.

Corollary 1. *If* $(A, \bullet_1, \ldots, \bullet_n, e_1, \ldots, e_n) \in \mathcal{V}_{n0}$ *then* $(\{f_a \,|\, a \in A\}, \circ_1, \ldots, \circ_n)$ *is isomorphic to* $(A, \bullet_1, \ldots, \bullet_n)$*, i.e. it is a functional representation of* $(A, \bullet_1, \ldots, \bullet_n)$*.*

Corollary 2. *In the case* $n = 1$ *Theorem 5 implies the Cayley Theorem for monoids.*

Corollary 3. *In the case* $n = 2$ *Theorem 5 implies the Cayley Theorem for bounded distributive lattices.*

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