Functorial equivalences

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The basic problem

(∗) Let $\mathcal{C}, \mathcal{D}$ be the categories, let $\mathcal{C}$ and let there be a functor $R : \mathcal{C} \to \mathcal{D}$.
Let $A, B \in \text{ob}(\mathcal{C})$,
$R(A) \cong R(B) \stackrel{?}{\Rightarrow} A \cong B$.

We can translate this problem using the category of R-pairs - the category of "possible counterexamples".

$R - pair =$ objects: $(M, N; t)$ such that $M, N \in \text{ob}(\mathcal{C})$,
$t : R(M) \to R(N)$ is an isomorphism
morphisms: pairs of morphisms between
the object components compatible with the isos.

Universal pairs

Now we can reformulate the problem (we call it *pair version* of the problem).
$(**)$ Is there an object in $R-pair$ with nonisomorphic object components?

If there is such, then we call it a *solution*. Exploring the category $R-pair$ will help us to find it, if it exists. Consider the forgetful functor $R-pair \to \mathcal{C}$,$(M,N;t) \mapsto M$. Now we can ask for existence of the free objects. If they exist, they give us a rough description of the objects. Therefore we focus on them and we call them *universal pairs*.

At first we show how to find the universal pair for a chosen instance of the problem using the algebraic methods.

Instances of the problem

1. Let C be a set, A, B be the sets. $\mathsf{hom}(C, A) \cong \mathsf{hom}(C, B) \stackrel{?}{\Rightarrow} A \cong B$

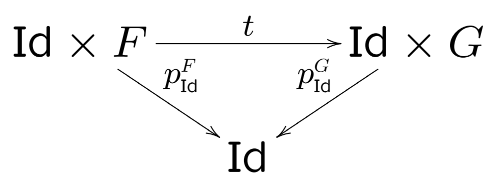
2. $F, G : \mathcal{S}et \to \mathcal{S}et$,

$$F \times F \cong G \times G \quad \stackrel{?}{\Rightarrow} \quad F \cong G$$

   Considering a functor $H_2 : \mathcal{S}et^{\mathcal{S}et} \to \mathcal{S}et^{\mathcal{S}et}$ such that for every $P : \mathcal{S}et \to \mathcal{S}et$ holds $H_2(P) = P \times P$, and for every natural transformation $\phi : P \to Q$ between the set functors we have $H_2(\phi) = (\phi, \phi)$, then the question is:

$$H_2(F) \cong H_2(G) \quad \stackrel{?}{\Rightarrow} \quad F \cong G$$

3. $F, G : \mathcal{S}et \to \mathcal{S}et$, let there be an isomorphism $t : \mathsf{Id} \times F \to \mathsf{Id} \times G$ such that the diagram commutes.



Does it imply $F \cong G$?

How to solve the problems?

The problem 1 is easy to solve. Since $\hom(C, A) = A^C$, then the answer depends on the cardinality of C. If C is nonempty finite set, then the answer is positive and its pair version has empty solution. If C is empty or infinite, then the answer is negative, the solution contains e.g. the pair $(2, 3; \gamma)$, where $\gamma : 2^C \to 3^C$ is an isomorphism.

The problem 2 is similar to 1 for the case $C = 2$, but it is extended up to the category of set functors. More precisely $H_2(F) = \hom(2, -) \circ F$. To solve it is much more difficult. In fact, during this talk we will not find the answer. We can just approach to it using the universal pair.

The problem 3 is actually instance of $(*)$ such that $R : \mathcal{S}et^{\mathcal{S}et} \to \mathcal{W}$,

$\mathcal{W} =$ objects: (M, m), $M : \mathcal{S}et \to \mathcal{S}et$, $m : M \to \mathrm{Id}$

morphisms: the natural transformations between functor-components compatible with the transformations to Id

$R(F) = (\mathrm{Id} \times F, p^F_{\mathrm{Id}})$. In this case we will find the answer. We show the result at the end of this presentation.

Construction of the universal pair for the problem (2)

Since $H_2(F) = \hom(2, -) \circ F$, it will be sufficient to find the universal pair for the problem 1, where $C = 2$. Let there be two sets A, B, such that there is isomorphism $t : A^2 \to B^2$. Therefore there are mappings $t_0, t_1 : A^2 \to B$,$t'_0, t'_1 : B^2 \to A$ satisfying for $a, b \in A, c, d \in B$

$$(a,b) \overset{t}{\mapsto} (t_0(a,b), t_1(a,b)) \overset{t^{-1}}{\mapsto} (t'_0(t_0(a,b), t_1(a,b)), t'_1(t_0(a,b), t_1(a,b)))$$

$$(c,d) \overset{t^{-1}}{\mapsto} (t'_0(c,d), t'_1(c,d)) \overset{t}{\mapsto} (t_0(t'_0(c,d), t'_1(c,d)), t_1(t'_0(c,d), t'_1(c,d)))$$

Therefore (A, B) has a structure of 2-sorted algebra with the signature $\Sigma = \{s_0, s_1, s'_0, s'_1\}$, $s_0, s_1 : (0,0) \to 1$, $s'_0, s'_1 : (1,1) \to 0$, where t_0, t_1, t'_0, t'_1, respectively, are the evaluations of the operation symbols.
Since t^{-1} and t are mutually inverse, the algebra satisfies the *duality identities*:

$$s'_0(s_0(x,y), s_1(x,y)) = x \quad s'_1(s_0(x,y), s_1(x,y)) = y$$

$$s_0(s'_0(u,v), s'_1(u,v)) = u \quad s_1(s'_0(u,v), s'_1(u,v)) = v.$$

Conversely, the duality identities yield the isomorphism between the second powers of the supports. Such an algebra will be called *square-iso algebra*.

The many-sorted algebras behave similarly as the one-sorted, e.g. for every set A and for a chosen item i of the list of algebra supports we can find a *free many-sorted algebra* over A in the i-labeled support such that A maps canonically to its i-labeled support.

Let A be a set, $\mathcal{A}$ be the free square-iso algebra over A in the 0-labeled support, $(P(A), Q(A))$ be the carrier of $\mathcal{A}$. Then $P(A)$ is a set of all the "correctly" composed terms in language of Σ with the variables from A and with the most outer symbol being a variable or s_0' or s_1' factorized over the duality identities. $Q(A)$ differs from $P(A)$ only in the most outer symbols, here these are s_0, or s_1.

The sets $P(A)$, $Q(A)$ are defined functorially, hence we have the functors $P, Q : \mathcal{S}et \to \mathcal{S}et$. Since $\mathcal{A}$ is a square-iso algebra, there is an isomorphism $\tau_A : P(A)^2 \to Q(A)^2$ which gives a rise to the natural transformation $\tau : hom(2, -) \circ P \to hom(2, -) \circ Q$. Therefore $hom(2, -) \circ P \cong hom(2, -) \circ Q$ and $(P \circ F, Q \circ F; \tau F)$ is a H_2-pair for every F. The freeness is given by $(P(A)$, $Q(A))$ being the carrier of a free algebra. But still do not know, if $P \ncong Q$.

Adjunction

All three shown instances of the basic problem actually have the same property: the category $\mathcal{C}$ has all colimits and functor R is a *right adjoint* and preserves the directed colimits (in (1) only if C is finite).

Recall that $L \dashv R : (\eta, \epsilon) : \mathcal{C} \to \mathcal{D}$ is the adjunction (L is a left adjoint to R and R is a right adjoint to L) iff
$R : \mathcal{C} \to \mathcal{D}$, $L : \mathcal{D} \to \mathcal{C}$ are the functors, $\eta : \mathrm{Id}_{\mathcal{D}} \to RL$, $\epsilon : LR \to \mathrm{Id}_{\mathcal{C}}$ are the natural transformations such that there is one-to-one correspondence between the morphisms $f : A \to R(B)$ in $\mathcal{D}$ and $\tilde{f} : L(A) \to B$ in $\mathcal{C}$, namely

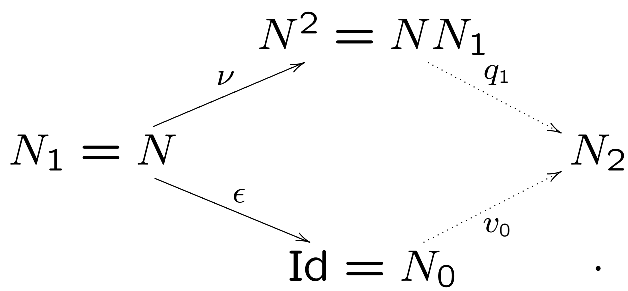
The examples of adjunction appear almost in every part of mathematics, e.g. $(C \times -, hom(C, -))$ on $\mathcal{S}et$, $(P \otimes -, hom_R(P, -))$ on modules over a commutative ring R, $(Free_\tau, Under_\tau)$ for a type τ and many similar cases of this kind, $(reflexion, embedding)$ of the reflexive subcategory, etc.

Theorem 1 *Let $L \dashv R : (\eta, \epsilon) : \mathcal{C} \to \mathcal{D}$ be the adjunction such that $\mathcal{C}$ has all colimits and R preserves the directed colimits. Then the category of R-pairs has the free objects.*

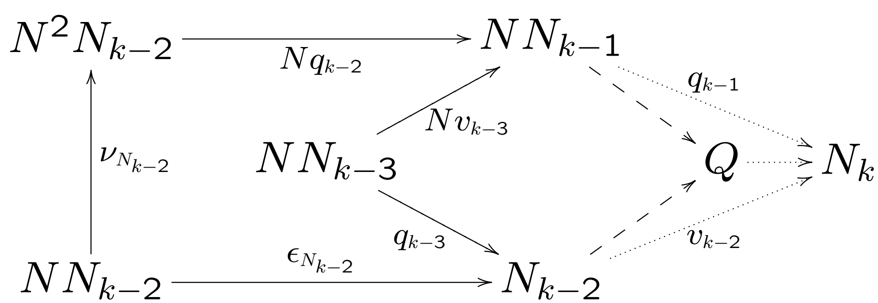
The proof, which will not be fully shown here, is constructive and we sketch the construction.

Construction of the free R-pair.

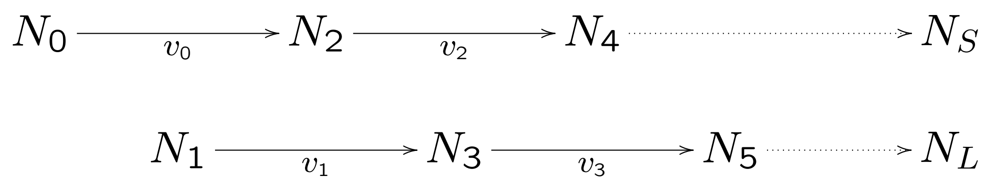
The adjunction yields the *comonad* (N, ϵ, ν) such that $N = L \circ R$ and $\nu = L\eta R : N \to N^2$, $\epsilon : N \to \mathrm{Id}$ are natural transformations given by adjunction. Then we can construct the following diagrams. Let $N_0 = \mathrm{Id}$, $N_1 = N$ and let $q_0 : NN_0 \to N_1$ be the identity on N. We define N_2 as a pushout of ϵ and ν, i.e.



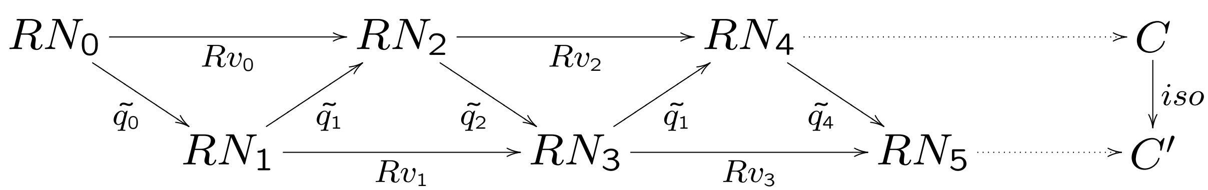
Since we already know $N_0, N_1, N_2, q_0, q_1, v_0$ we define recursively for $n \in \mathbf{N}$, $n \geq 3$ the object N_n and the morphisms $q_{n-1} : NN_{n-1} \to N_n, v_n : N_{n-2} \to N_n$ as the colimit of the diagram drawn by solid lines:



We define the functors N_S and N_L as the directed colimits in of the chains $\mathcal{C}$:



One can prove, that for every $k \in \omega$ holds $Rv_k = \widetilde{q_{k+1}} \circ \tilde{q}_k$. Then R-images of these chains have the same colimit C.



If R preserves the directed colimits, then $RN_S \cong C \cong RN_L$. Therefore there is an isomorphism $\tau : RN_S \to RN_L$ and $(N_S, N_L; \tau)$ is an R-pair.

Construction of the universal pair for the problem (3)

This procedure shown above can be used to find the universal pairs for every instance of problem $(*)$. In case of (3),

$$(N_S, N_L; \tau) \cong (Mon_S \times F, Mon_L \times F; \tau),$$

where $Mon_S(A), Mon_L(A)$ are the subsets of the free monoid over a set A satisfying literally (i.e. on the variables) the equation

$$aa = 1,$$

namely $Mon_S(A)$ and $Mon_L(A)$ contains all terms of the even and odd "length", respectively.
The transformation $\tau_F : \mathrm{Id} \times Mon_S \times F \to \mathrm{Id} \times Mon_S \times F$ is actually a pair of transformations $\tau_F = (\sigma, \mathrm{id}_F)$, where $\sigma : \mathrm{Id} \times Mon_S \to \mathrm{Id} \times Mon_S$ is an isomorphism defined on a set A as follows:

$$\tau_A(a, x) = (a, ax),\ \tau_A^{-1}(a, y) = (a, ay).$$

Theorem 2 *One can prove that $Mon_S \ncong Mon_L$, i.e. the answer for the question 3 is negative.*