How subalgebra lattices of direct powers determine an algebra?

in particular

Clones of entropic algebras with unit

Dragan Mašulović and Reinhard Pöschel

University Novi Sad
and
Technische Universität Dresden

Summer School on General Algebra and Ordered Sets
Třešť, September 2008

Outline

Preliminary remark

This talk was inspired by papers of
K. Kearnes and A. Szendrei

- *Clones of finite groups.* Algebra Universalis 54 (2005), 23–52.
- *Groups with identical subgroup lattices in all powers.* J. Group Theory, 7 (2004), 385–402.
- *Clones of 2-step nilpotent groups.* Algebra Universalis

First results presented by R. Pöschel
at AAA74 (Tampere 2007) and AAA76 (Linz 2008)

Now generalization

Outline

Notations

$\mathcal{A} = \langle A, (f_i)_{i \in I} \rangle = \langle A, F \rangle$ algebra

$\mathrm{Sub}(\mathcal{A}^m)$ subalgebra lattice of the m-th direct power
($\mathrm{Inv}^{(m)} F$ m-ary *invariant* relations of F)

$\mathrm{Clo}(\mathcal{A})$ clone of term operations
$= \langle F \rangle_{O_A}$ clone generated by F in the clone $\mathrm{Op}(A)$ of all finitary operations on A

▸ Definition clone

weakly isomorphic algebras: $\exists \mathcal{A}' : \mathcal{A} \cong \mathcal{A}' \wedge \mathrm{Clo}(\mathcal{A}') = \mathrm{Clo}(\mathcal{B})$

$\mathcal{A} = \langle A, F \rangle$, $\mathcal{B} = \langle A, G \rangle$
term equivalent algebras: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$

Notations

$\mathcal{A} = \langle A, (f_i)_{i\in I}\rangle = \langle A, F\rangle$ algebra

$\mathrm{Sub}(\mathcal{A}^m)$ subalgebra lattice of the m-th direct power
($\mathrm{Inv}^{(m)} F$ m-ary *invariant* relations of F)

$\mathrm{Clo}(\mathcal{A})$ clone of term operations
$= \langle F\rangle_{O_A}$ clone generated by F in the clone $\mathrm{Op}(A)$ of all finitary operations on A

Definition clone

weakly isomorphic algebras: $\exists \mathcal{A}' : \mathcal{A} \cong \mathcal{A}' \wedge \mathrm{Clo}(\mathcal{A}') = \mathrm{Clo}(\mathcal{B})$

$\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$
term equivalent algebras: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$

Notations

$\mathcal{A} = \langle A, (f_i)_{i \in I} \rangle = \langle A, F \rangle$ algebra

$\mathrm{Sub}(\mathcal{A}^m)$ subalgebra lattice of the m-th direct power
($\mathrm{Inv}^{(m)} F$ m-ary *invariant* relations of F)

$\mathrm{Clo}(\mathcal{A})$ clone of term operations
$= \langle F \rangle_{O_A}$ clone generated by F in the clone $\mathrm{Op}(A)$ of all finitary operations on A

Definition clone

weakly isomorphic algebras: $\exists \mathcal{A}' : \mathcal{A} \cong \mathcal{A}' \wedge \mathrm{Clo}(\mathcal{A}') = \mathrm{Clo}(\mathcal{B})$

$\mathcal{A} = \langle A, F \rangle$, $\mathcal{B} = \langle A, G \rangle$
term equivalent algebras: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$

Notations

$\mathcal{A} = \langle A, (f_i)_{i \in I} \rangle = \langle A, F \rangle$ algebra

Sub($\mathcal{A}^m$) subalgebra lattice of the m-th direct power
(Inv$^{(m)}$ F m-ary *invariant* relations of F)

Clo($\mathcal{A}$) clone of term operations
$= \langle F \rangle_{O_A}$ clone generated by F in the clone Op(A) of all finitary operations on A

Definition clone

weakly isomorphic algebras: $\exists \mathcal{A}' : \mathcal{A} \cong \mathcal{A}' \wedge \text{Clo}(\mathcal{A}') = \text{Clo}(\mathcal{B})$

$\mathcal{A} = \langle A, F \rangle$, $\mathcal{B} = \langle A, G \rangle$
term equivalent algebras: $\text{Clo}(\mathcal{A}) = \text{Clo}(\mathcal{B})$

Notations

$\mathcal{A} = \langle A, (f_i)_{i\in I}\rangle = \langle A, F\rangle$ algebra

$\mathrm{Sub}(\mathcal{A}^m)$ subalgebra lattice of the m-th direct power
($\mathrm{Inv}^{(m)} F$ m-ary *invariant* relations of F)

$\mathrm{Clo}(\mathcal{A})$ clone of term operations
$= \langle F\rangle_{O_A}$ clone generated by F in the clone $\mathrm{Op}(A)$ of all finitary operations on A ▸ Definition clone

weakly isomorphic algebras: $\exists \mathcal{A}' : \mathcal{A} \cong \mathcal{A}' \wedge \mathrm{Clo}(\mathcal{A}') = \mathrm{Clo}(\mathcal{B})$

$\mathcal{A} = \langle A, F\rangle,\ \mathcal{B} = \langle A, G\rangle$
term equivalent algebras: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$

Problem: How $\mathrm{Sub}(\mathcal{A}^n)$ determines the algebra $\mathcal{A}$?

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

If $\mathcal{A}, \mathcal{B}$ have the same underlying set:

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

(for all $n \in \mathbb{N}$, or for a particular $n = n_0$)

Problem: How $\mathrm{Sub}(\mathcal{A}^n)$ determines the algebra $\mathcal{A}$?

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

If $\mathcal{A}, \mathcal{B}$ have the same underlying set:

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

(for all $n \in \mathbb{N}$, or for a particular $n = n_0$)

Problem: How $\mathrm{Sub}(\mathcal{A}^n)$ determines the algebra $\mathcal{A}$?

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

If $\mathcal{A}, \mathcal{B}$ have the same underlying set:

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

(for all $n \in \mathbb{N}$, or for a particular $n = n_0$)

Problem: How $\mathrm{Sub}(\mathcal{A}^n)$ determines the algebra $\mathcal{A}$?

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

If $\mathcal{A}, \mathcal{B}$ have the same underlying set:

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \implies ???$$

(for all $n \in \mathbb{N}$, or for a particular $n = n_0$)

e.g., $\mathcal{A}, \mathcal{B} \in \mathcal{K}$

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \cong \mathrm{Clo}(\mathcal{B}) \text{ or } \mathcal{A} \cong \mathcal{B}$$

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

$\mathrm{Clo}(\mathcal{A})$ is unique in $\mathcal{K}$ modulo n-ary invariants

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathcal{A} = \mathcal{B}$$

$\mathcal{A}$ is unique in $\mathcal{K}$ modulo n-ary invariants

And if so, does n depend on the cardinality of $\mathcal{A}$?

unique modulo invariants $:\Longleftrightarrow$ $[\forall n : \mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n)] \overset{?}{\Longrightarrow} \ldots$

e.g., $\mathcal{A}, \mathcal{B} \in \mathcal{K}$

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \cong \mathrm{Clo}(\mathcal{B}) \text{ or } \mathcal{A} \cong \mathcal{B}$$

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

$\mathrm{Clo}(\mathcal{A})$ is unique in $\mathcal{K}$ modulo n-ary invariants

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathcal{A} = \mathcal{B}$$

$\mathcal{A}$ is unique in $\mathcal{K}$ modulo n-ary invariants

And if so, does n depend on the cardinality of $\mathcal{A}$?

unique modulo invariants $:\Longleftrightarrow$ $[\forall n : \mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n)] \overset{?}{\Longrightarrow} \ldots$

e.g., $\mathcal{A}, \mathcal{B} \in \mathcal{K}$

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \cong \mathrm{Clo}(\mathcal{B}) \text{ or } \mathcal{A} \cong \mathcal{B}$$

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

$\mathrm{Clo}(\mathcal{A})$ is unique in $\mathcal{K}$ modulo n-ary invariants

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathcal{A} = \mathcal{B}$$

$\mathcal{A}$ is unique in $\mathcal{K}$ modulo n-ary invariants

And if so, does n depend on the cardinality of $\mathcal{A}$?

unique modulo invariants : $\Longleftrightarrow$ $[\forall n : \mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n)] \overset{?}{\Longrightarrow} \ldots$

e.g., $\mathcal{A}, \mathcal{B} \in \mathcal{K}$

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \cong \mathrm{Clo}(\mathcal{B}) \text{ or } \mathcal{A} \cong \mathcal{B}$$

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

$\mathrm{Clo}(\mathcal{A})$ is unique in $\mathcal{K}$ modulo n-ary invariants

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathcal{A} = \mathcal{B}$$

$\mathcal{A}$ is unique in $\mathcal{K}$ modulo n-ary invariants

And if so, does n depend on the cardinality of $\mathcal{A}$?

unique modulo invariants : $\Longleftrightarrow$ $[\forall n : \mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n)] \overset{?}{\Longrightarrow} \ldots$

e.g., $\mathcal{A}, \mathcal{B} \in \mathcal{K}$

$$\mathrm{Sub}(\mathcal{A}^n) \cong \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \cong \mathrm{Clo}(\mathcal{B}) \text{ or } \mathcal{A} \cong \mathcal{B}$$

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

$\mathrm{Clo}(\mathcal{A})$ is unique in $\mathcal{K}$ modulo n-ary invariants

$$\mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n) \overset{?}{\Longrightarrow} \mathcal{A} = \mathcal{B}$$

$\mathcal{A}$ is unique in $\mathcal{K}$ modulo n-ary invariants

And if so, does n depend on the cardinality of $\mathcal{A}$?

unique modulo invariants $:\Longleftrightarrow$ $[\forall n : \mathrm{Sub}(\mathcal{A}^n) = \mathrm{Sub}(\mathcal{B}^n)] \overset{?}{\Longrightarrow} \dots$

Outline

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \rhd \varrho$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \rhd \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \rhd \varrho\}$ polymorphisms

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \triangleright \varrho$

$$
\begin{array}{l}
f(\;a_{11}\quad a_{12}\quad \dots\quad a_{1n}\;) = \bullet \\
f(\;a_{21}\quad a_{22}\quad \dots\quad a_{2n}\;) = \bullet \\
\vdots \\
f(\;a_{m1}\quad a_{m2}\quad \dots\quad a_{mn}\;) = \bullet
\end{array}
$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)

$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$ polymorphisms

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \rhd \varrho$

$$
\begin{array}{lllllll}
f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & \vdots & \vdots & & \vdots & & \vdots \\
f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \in \varrho & \in \varrho & \dots & \in \varrho & &
\end{array}
$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \rhd \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \rhd \varrho\}$ polymorphisms

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \rhd \varrho$

$$
\begin{array}{lcccccc}
f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & & & & & & \\
f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \in \varrho & \in \varrho & \dots & \in \varrho & \Rightarrow & \in \varrho
\end{array}
$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \rhd \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \rhd \varrho\}$ polymorphisms

TECHNISCHE UNIVERSITÄT DRESDEN

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \triangleright \varrho$

$$
\begin{array}{ccccccc}
f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & & & & & & \\
f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \in \varrho & \in \varrho & \dots & \in \varrho & \Rightarrow & \in \varrho
\end{array}
$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$ polymorphisms

The Galois connection Pol – Inv

induced by the relation **function f preserves relation ϱ**: $f \triangleright \varrho$

$$
\begin{array}{rcccccl}
f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & & & & & & \\
f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \in \varrho & \in \varrho & \dots & \in \varrho & \Rightarrow & \in \varrho
\end{array}
$$

$F \subseteq \mathrm{Op}(A)$ (set of all finitary operations $f : A^n \to A$)
$Q \subseteq \mathrm{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$\mathrm{Inv}\, F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$ invariant relations

$\mathrm{Pol}\, Q := \{f \in \mathrm{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$ polymorphisms

Special case: commuting operations

Operations $f : A^n \to A$ and $g : A^m \to A$ commute if

“$f(g(X)) = g(f(X))$”

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{cccc} g & g & \cdots & g \end{array} \\
\begin{array}{c} f \\ f \\ \vdots \\ f \end{array} &
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \vdots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\end{array}
=
\begin{array}{c} g \\ \begin{pmatrix} \circ \\ \circ \\ \vdots \\ \circ \end{pmatrix} \end{array} \\
\begin{array}{cccc} \downarrow & \downarrow & & \downarrow \end{array} \qquad \downarrow \\
f\begin{pmatrix} \circ & \circ & \cdots & \circ \end{pmatrix} = \square
\end{array}
$$

Then

$$f, g \text{ commute} \iff f \triangleright g^\bullet \iff g \triangleright f^\bullet$$

where $f^\bullet := \{(a_1, \ldots, a_n, b) \in A^{n+1} \mid f(a_1, \ldots, a_n) = b\}$ is the graph of f

Special case: commuting operations

Operations $f : A^n \to A$ and $g : A^m \to A$ commute if

"$f(g(X)) = g(f(X))$"

$$
\begin{array}{c}
\begin{matrix} & g & g & \cdots & g & & & g \\
\begin{matrix} f \\ f \\ \vdots \\ f \end{matrix} & \left(\begin{matrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{matrix}\right) & & & & = & \left(\begin{matrix} \circ \\ \circ \\ \vdots \\ \circ \end{matrix}\right) \end{matrix} \\
\downarrow \quad \downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
f \left(\circ \quad \circ \quad \cdots \quad \circ \right) = \square
\end{array}
$$

Then

$$f, g \text{ commute} \iff f \triangleright g^\bullet \iff g \triangleright f^\bullet$$

where $f^\bullet := \{(a_1, \ldots, a_n, b) \in A^{n+1} \mid f(a_1, \ldots, a_n) = b\}$ is the graph of f

Special case: commuting operations

Operations $f : A^n \to A$ and $g : A^m \to A$ **commute** if

$$\text{“}f(g(X)) = g(f(X))\text{”} \qquad \begin{array}{c} \\ f \\ f \\ \vdots \\ f \\ \\ f \end{array} \begin{array}{cccc} g & g & \cdots & g \\ \left(\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{array}\right) \\ \downarrow & \downarrow & & \downarrow \\ (\circ & \circ & \cdots & \circ) \end{array} \begin{array}{c} g \\ = \left(\begin{array}{c} \circ \\ \circ \\ \vdots \\ \circ \end{array}\right) \\ \downarrow \\ = \quad \square \end{array}$$

Then

$$f, g \text{ commute} \iff f \rhd g^{\bullet} \iff g \rhd f^{\bullet}$$

where $f^{\bullet} := \{(a_1, \dots, a_n, b) \in A^{n+1} \mid f(a_1, \dots, a_n) = b\}$ is the **graph** of f

Example ($f = g,\ n = m = 2$)

A binary operation $+ : A^2 \to A$ commutes with itself ($+ \triangleright +^\bullet$) iff

$$\forall a, b, c, d \in A : (a+b)+(c+d) = (a+c)+(b+d)\,.$$

In particular, every commutative operation commutes with itself.

Theorem (Characterization of Galois closed elements)

$\mathcal{A} = \langle A, F\rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- $m\text{-}\mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$
  If $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$:
  $\mathrm{Clo}(\mathcal{A})$ determined by its m-ary invariants,
  m-locally closed clone, clone with m-interpolation property)
- $[Q] = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F\rangle$ *(arbitrary) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*

[1] Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин

Theorem (Characterization of Galois closed elements)

$\mathcal{A} = \langle A, F \rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F \rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- m-$\mathrm{Loc}\langle F \rangle = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$
  If $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$:
  $\mathrm{Clo}(\mathcal{A})$ determined by its m-ary invariants,
  m-locally closed clone, clone with m-interpolation property)
- $[Q] = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F \rangle$ *(arbitrary) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F \rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*

[1] Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин

Theorem (Characterization of Galois closed elements)

$\mathcal{A} = \langle A, F\rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- $m\text{-}\mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$
  If $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$:
  $\mathrm{Clo}(\mathcal{A})$ determined by its *m*-ary invariants,
  m-locally closed clone, clone with *m*-interpolation property)
- $[Q] = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F\rangle$ *(arbitrary) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*

[1] Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин

Theorem (Characterization of Galois closed elements)

$\mathcal{A} = \langle A, F\rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- $m\text{-}\mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$
  If $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$:
  $\mathrm{Clo}(\mathcal{A})$ determined by its m-ary invariants,
  m-locally closed clone, clone with m-interpolation property)
- $[Q] = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F\rangle$ *(arbitrary) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*

[1] Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин

Theorem (Characterization of Galois closed elements)

$\mathcal{A} = \langle A, F\rangle$ *finite algebra.*

- $\mathrm{Clo}(\mathcal{A}) = \langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(clone generated by F)* [1],
- $m\text{-}\mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$
  If $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)}\,F$:
  $\mathrm{Clo}(\mathcal{A})$ determined by its m-ary invariants,
  m-locally closed clone, clone with m-interpolation property)
- $[Q] = \mathrm{Inv}\,\mathrm{Pol}\,Q$ *(relational clone generated by Q).*

$\mathcal{A} = \langle A, F\rangle$ *(arbitrary) algebra*

- $\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\langle F\rangle = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(locally closed clone generated by F)*

[1] Lev Arkadevic Kalužnin, Лев Аркадевич Калужнин

Immediate consequence: An "answer" to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e.* $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\, F$ *(in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$*)*
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\, F = \mathrm{Inv}\, G$. Then

$$\begin{aligned}\mathrm{Clo}(\mathcal{A}) &\overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\, F = \mathrm{Pol}\,\mathrm{Inv}\, G \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B}) \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}).\end{aligned}$$

$\square$

Immediate consequence: An "answer" to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F \Longrightarrow \mathrm{Clo}(\mathcal{A})$ unique modulo invariants (in $\mathcal{K}$)

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e. $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F$ (in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$)
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\begin{aligned}\mathrm{Clo}(\mathcal{A}) &\overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F = \mathrm{Pol}\,\mathrm{Inv}\,G \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B}) \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}). \quad \square\end{aligned}$$

Immediate consequence: An "answer" to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F \overset{?\not\Leftarrow!}{\Longrightarrow} \mathrm{Clo}(\mathcal{A})$ unique modulo invariants (in $\mathcal{K}$)

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e. $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F$ (in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$)
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\mathrm{Clo}(\mathcal{A}) \overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F = \mathrm{Pol}\,\mathrm{Inv}\,G$$
$$= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B})$$
$$= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}). \quad \square$$

Immediate consequence: An “answer” to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F \overset{?\not\Leftarrow!}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \text{ unique modulo invariants (in } \mathcal{K})$$

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e.* $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F$ *(in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$*)*
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\begin{aligned}\mathrm{Clo}(\mathcal{A}) &\overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F = \mathrm{Pol}\,\mathrm{Inv}\,G \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B}) \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}).\end{aligned}$$

$\square$

Immediate consequence: An "answer" to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F \overset{?\not\Leftarrow!}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \text{ unique modulo invariants (in } \mathcal{K})$$

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e. $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F$ (in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$)
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\mathrm{Clo}(\mathcal{A}) \overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F = \mathrm{Pol}\,\mathrm{Inv}\,G$$
$$= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B})$$
$$= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}). \quad \square$$

Immediate consequence: An "answer" to our problem

locally closed $\Longrightarrow$ clone unique modulo invariants (in $\mathcal{K}$)

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F \overset{?\not\Leftarrow!}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \text{ unique modulo invariants (in } \mathcal{K})$$

Proposition. *Let $\mathcal{K}$ be the class of all algebras with a locally closed clone of term operations, i.e. $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F$ (in particular, $\mathcal{K}$ contains all finite algebras).*
Then the clone of every $\mathcal{A} \in \mathcal{K}$ is unique in $\mathcal{K}$
(as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$)
modulo invariants.

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\begin{aligned}\mathrm{Clo}(\mathcal{A}) &\overset{\mathcal{A}\in\mathcal{K}}{=} \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}\,F = \mathrm{Pol}\,\mathrm{Inv}\,G \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B}) \\ &= \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}).\end{aligned}$$

$\square$

TECHNISCHE UNIVERSITÄT DRESDEN

Immediate consequence: An "answer" to our problem

m-locally closed $\Longrightarrow$ clone unique modulo *m*-ary invariants (in $\mathcal{K}$)

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)} F \overset{?\not\Leftarrow!}{\Longrightarrow} \mathrm{Clo}(\mathcal{A}) \text{ unique modulo } m\text{-ary invariants (in } \mathcal{K}$$

Proposition. *Let $\mathcal{K}_m$ be the class of all algebras with an m-locally closed clone of term operations, i.e. $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(m)} F$ (in particular, $\mathcal{K}$ contains all m-locally closed finite algebras). Then the clone of every $\mathcal{A} \in \mathcal{K}_m$ is unique in $\mathcal{K}$ (as well as in the class $\mathcal{K}_{\mathcal{A}}$ of all algebras $\mathcal{B}$ with $\mathrm{Clo}(\mathcal{A}) \subseteq \mathrm{Clo}(\mathcal{B})$) modulo m-ary invariants.*

Proof. Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle A, G\rangle$ and $\mathrm{Inv}\,F = \mathrm{Inv}\,G$. Then

$$\begin{aligned}\mathrm{Clo}(\mathcal{A}) \overset{\mathcal{A}\in\mathcal{K}}{=} m\text{-}\mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) &= \mathrm{Pol}\,\mathrm{Inv}^{(m)} F = \mathrm{Pol}\,\mathrm{Inv}^{(m)} G \\ &= m\text{-}\mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}}{=} \mathrm{Clo}(\mathcal{B}) \\ &= m\text{-}\mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \supseteq \mathrm{Clo}(\mathcal{B}) \overset{\mathcal{B}\in\mathcal{K}_{\mathcal{A}}}{\supseteq} \mathrm{Clo}(\mathcal{A}). \quad \square\end{aligned}$$

Outline

Near unanimity term operation

Theorem

Let $\mathcal{A} = \langle A, F \rangle$ be a finite algebra such that there is a $(d+1)$-ary near unanimity term operation. Then $\operatorname{Clo} \mathcal{A} = \langle F \rangle = \operatorname{Pol} \operatorname{Inv}^{(d)} F$ (in particular, $\operatorname{Clo}(\mathcal{A})$ is unique in $\mathcal{K}_d$ modulo d-ary invariants).

near unanimity operation $f : A^n \to A$:

$$
\begin{aligned}
f(x, y, \ldots, y) &= y \\
f(y, x, \ldots, y) &= y \\
&\vdots \\
f(y, y, \ldots, x) &= y
\end{aligned}
$$

Generalization: k-edge term operation

Kearnes/Szendrei (personal communication May 2007)

Theorem

Let $\mathcal{A} = \langle A, F\rangle$ be a finite algebra such that

(1) *$\mathcal{A}$ has a k-edge term for some $k \in \mathbb{N}$,*

(2) *$\mathcal{A}$ generates a residually small variety.*

Then $\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(d)}\,F$ for some d (depending only on the cardinality $|A|$).

k-edge term

Generalization of both, near-unanimity term and Mal'cev term

A *k-edge term* is a $(k+1)$-ary term satisfying the identities:

$$
\begin{aligned}
e(x,x,y,y,y,\dots,y,y) &= y\\
e(x,y,x,y,y,\dots,y,y) &= y\\
e(y,y,y,x,y,\dots,y,y) &= y\\
e(y,y,y,y,x,\dots,y,y) &= y\\
&\dots\\
e(y,y,y,y,y,\dots,y,x) &= y
\end{aligned}
$$

(introduced by J. Berman, P. Idziak, P. Markovic, R. McKenzie, M. Valeriote, R. Willard: *Tractability and learnability arising from algebras with few subpowers,* Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science, 2007)

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \implies G \cong H$?

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$$

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$$

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$$

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$$

But for abelian groups we have (R. Baer, 1939):

$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):

$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:

$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Is every group determined (up to isomorphism) by the subgroup lattices of its finite powers?

Answer (Kearnes/Szendrei):

$$[\forall n \in \mathbb{N}\ \mathrm{Sub}(G^n) \cong \mathrm{Sub}(H^n)] \overset{\text{No}}{\Longrightarrow} G \cong H$$

But for abelian groups we have (R. Baer, 1939):
$\mathrm{Sub}(G^3) \cong \mathrm{Sub}(H^3) \implies G \cong H$

moreover, $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies G = H$ (diploma thesis H.A. Pham '07)

More general (Kearnes/Szendrei):
$G = \langle A, \cdot \rangle$, $H = \langle A, \odot \rangle$ finite groups with abelian Sylow subgroups:

$\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3) \implies \mathrm{Clo}(G) = \mathrm{Clo}(H)$ (term equivalent)

cyclic Sylow subgroups:
$\mathrm{Sub}(G^2) \cong \mathrm{Sub}(H^2) \implies G, H$ weakly isomorphic

Theorem (J.W.Snow): There is no k such that for every finite group G, $\mathrm{Clo}(G)$ is determined by the k-ary invariants of G.

Outline

Entropic algebras

An algebra $\mathcal{A} = \langle A, F \rangle$ is called *entropic* if every two operations $f, g \in F$ commute (i.e. $\forall f, g \in F : f \triangleright g^\bullet$)

$e \in A$ *weakly neutral* for an operation $f : A^n \to A : \iff$ $n \geq 2$ and $\forall i \in \{1, \ldots, n\} \forall x \in A : f(e, \ldots, e, x, e, \ldots, e) = x$ (x at i-th place)

$\mathcal{A} = \langle A, F \rangle$ *entropic with weakly neutral element* $:\iff$ $\mathcal{A}$ entropic and $\exists\, e \in A : e$ is weakly neutral for every $f \in F$ (consequently all operations $f \in F$ have arity at least 2)

Examples: commutative monoids $\langle A, \cdot, e \rangle$ (in particular abelian groups)

Entropic algebras

An algebra $\mathcal{A} = \langle A, F \rangle$ is called *entropic* if every two operations $f, g \in F$ commute (i.e. $\forall f, g \in F : f \triangleright g^\bullet$)

$e \in A$ *weakly neutral* for an operation $f : A^n \to A :\Longleftrightarrow$
$n \geq 2$ and $\forall i \in \{1, \ldots, n\} \forall x \in A : f(e, \ldots, e, x, e, \ldots, e) = x$
(x at i-th place)

$\mathcal{A} = \langle A, F \rangle$ *entropic with weakly neutral element*
$:\Longleftrightarrow$ $\mathcal{A}$ entropic and $\exists\, e \in A : e$ is weakly neutral for every $f \in F$
(consequently all operations $f \in F$ have arity at least 2)

Examples: commutative monoids $\langle A, \cdot, e \rangle$ (in particular abelian groups)

Entropic algebras

An algebra $\mathcal{A} = \langle A, F \rangle$ is called *entropic* if every two operations $f, g \in F$ commute (i.e. $\forall f, g \in F : f \triangleright g^{\bullet}$)

$e \in A$ *weakly neutral* for an operation $f : A^n \to A :\iff$
$n \geq 2$ and $\forall i \in \{1, \ldots, n\} \forall x \in A : f(e, \ldots, e, x, e, \ldots, e) = x$
(x at i-th place)

$\mathcal{A} = \langle A, F \rangle$ *entropic with weakly neutral element*
$:\iff$ $\mathcal{A}$ entropic and $\exists\, e \in A : e$ is weakly neutral for every $f \in F$
(consequently all operations $f \in F$ have arity at least 2)

Examples: commutative monoids $\langle A, \cdot, e \rangle$ (in particular abelian groups)

Entropic algebras

An algebra $\mathcal{A} = \langle A, F\rangle$ is called *entropic* if every two operations $f, g \in F$ commute (i.e. $\forall f, g \in F : f \rhd g^{\bullet}$)

$e \in A$ *weakly neutral* for an operation $f : A^n \to A :\iff$
$n \geq 2$ and $\forall i \in \{1, \ldots, n\} \forall x \in A : f(e, \ldots, e, x, e, \ldots, e) = x$
(x at i-th place)

$\mathcal{A} = \langle A, F\rangle$ *entropic with weakly neutral element*
$:\iff$ $\mathcal{A}$ entropic and $\exists\, e \in A : e$ is weakly neutral for every $f \in F$
(consequently all operations $f \in F$ have arity at least 2)

Examples: commutative monoids $\langle A, \cdot, e\rangle$ (in particular abelian groups)

Entropic algebras and monoids

Lemma

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Take any $f \in F$ (! arity of f is at least 2) and define

$$x \cdot y := f(x, y, e, \dots, e),$$

Then $\mathcal{M} = \langle A, \cdot, e \rangle$ is a commutative monoid (called monoid associated to $\mathcal{A}$).

Theorem

Let $\mathcal{A} = \langle A, F, e \rangle$ be an entropic algebra with weakly neutral element e. Then $\mathcal{A}$ is term-equivalent to any associated monoid $\mathcal{M} = \langle A, \cdot, e \rangle$: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{M})$.

idea of the proof: For m-ary $g \in F$ show

$$g(x_1, x_2, \dots, x_m) = x_1 \cdot x_2 \cdot \dots \cdot x_m.$$

Entropic algebras and monoids

Lemma

Let $\mathcal{A} = \langle A, F, e\rangle$ be an entropic algebra with weakly neutral element e. Take any $f \in F$ (! arity of f is at least 2) and define

$$x \cdot y := f(x, y, e, \dots, e),$$

Then $\mathcal{M} = \langle A, \cdot, e\rangle$ is a commutative monoid (called monoid associated to $\mathcal{A}$).

Theorem

Let $\mathcal{A} = \langle A, F, e\rangle$ be an entropic algebra with weakly neutral element e. Then $\mathcal{A}$ is term-equivalent to any associated monoid $\mathcal{M} = \langle A, \cdot, e\rangle$: $\mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{M})$.

idea of the proof: For m-ary $g \in F$ show

$$g(x_1, x_2, \dots, x_m) = x_1 \cdot x_2 \cdot \dots \cdot x_m.$$

Crucial Lemma for the proof

Lemma

Let $f : A^n \to A$ and $g : A^m \to A$ be operations on A with $m \leq n$ such that f and g commute and have a common weakly neutral element e (thus $2 \leq m \leq n$). Then

$$g(x_1, \ldots, x_m) = f(x_1, \ldots, x_m, e, \ldots, e)$$

(in particular $f = g$ for $m = n$).

Proof.

$$\begin{array}{c} \\ f \\ f \\ \vdots \\ f \end{array} \begin{array}{ccccccc} g & g & \cdots & g & g & \cdots & g \\ \end{array}$$

$$\begin{array}{c} f \\ f \\ \vdots \\ f \end{array} \begin{pmatrix} x_1 & e & \ldots & e & e & \ldots & e \\ e & x_2 & \ldots & e & e & \ldots & e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e & e & \ldots & x_m & e & \ldots & e \end{pmatrix} = \overset{g}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}}$$

implies $f(x_1, x_2, \ldots, x_m, e, \ldots, e) = g(x_1, x_2, \ldots, x_m)$ □

Crucial Lemma for the proof

Lemma

Let $f : A^n \to A$ and $g : A^m \to A$ be operations on A with $m \leq n$ such that f and g commute and have a common weakly neutral element e (thus $2 \leq m \leq n$). Then

$$g(x_1, \ldots, x_m) = f(x_1, \ldots, x_m, e, \ldots, e)$$

(in particular $f = g$ for $m = n$).

Proof.

$$\begin{array}{c} \\ f \\ f \\ \vdots \\ f \end{array} \begin{array}{cccccccc} g & g & \ldots & g & g & \ldots & g \\ \end{array}$$

$$\begin{array}{c} f \\ f \\ \vdots \\ f \end{array} \begin{pmatrix} x_1 & e & \ldots & e & e & \ldots & e \\ e & x_2 & \ldots & e & e & \ldots & e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e & e & \ldots & x_m & e & \ldots & e \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}^{g}$$

implies $f(x_1, x_2, \ldots, x_m, e, \ldots, e) = g(x_1, x_2, \ldots, x_m)$ □

TECHNISCHE UNIVERSITÄT DRESDEN

Crucial Lemma for the proof

Lemma

Let $f : A^n \to A$ and $g : A^m \to A$ be operations on A with $m \leq n$ such that f and g commute and have a common weakly neutral element e (thus $2 \leq m \leq n$). Then

$$g(x_1, \dots, x_m) = f(x_1, \dots, x_m, e, \dots, e)$$

(in particular $f = g$ for $m = n$).

Proof.

$$\begin{array}{c} \\ f \\ f \\ \vdots \\ f \end{array}
\begin{array}{c}
\begin{array}{ccccccc} g & g & \dots & g & g & \dots & g \end{array} \\
\begin{pmatrix}
x_1 & e & \dots & e & e & \dots & e \\
e & x_2 & \dots & e & e & \dots & e \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e & e & \dots & x_m & e & \dots & e
\end{pmatrix}
\end{array}
=
\begin{array}{c} g \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \end{array}$$

implies $f(x_1, x_2, \dots, x_m, e, \dots, e) = g(x_1, x_2, \dots, x_m)$ □

Entropic algebras are unique modulo ternary invariants

Theorem

(1) *The clone of each entropic algebra $\mathcal{A} = \langle A, F, e \rangle$ with weakly neutral element is determined by its ternary invariants:*

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(3)} F.$$

(2) *Let $\mathcal{E}$ be the class of all entropic algebras with a weakly neutral element. Then the clone of each $\mathcal{A} \in \mathcal{E}$ is unique in $\mathcal{E}$ modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E},\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

(3) *Let $\mathcal{E}_n$ be the class of all entropic algebras $\mathcal{A} = \langle A, f, e \rangle$ with a weakly neutral element and one n-ary fundamental operation. Then each $\mathcal{A} \in \mathcal{E}_n$ is unique in $\mathcal{E}_n$ modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E}_n,\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathcal{A} = \mathcal{B}.$$

Entropic algebras are unique modulo ternary invariants

Theorem

(1) *The clone of each entropic algebra $\mathcal{A} = \langle A, F, e \rangle$ with weakly neutral element is determined by its ternary invariants:*

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\, F.$$

(2) *Let $\mathcal{E}$ be the class of all entropic algebras with a weakly neutral element. Then the clone of each $\mathcal{A} \in \mathcal{E}$ is unique in $\mathcal{E}$ modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E},\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

(3) *Let $\mathcal{E}_n$ be the class of all entropic algebras $\mathcal{A} = \langle A, f, e \rangle$ with a weakly neutral element and one n-ary fundamental operation. Then each $\mathcal{A} \in \mathcal{E}_n$ is unique in $\mathcal{E}_n$ modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E}_n,\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathcal{A} = \mathcal{B}.$$

Entropic algebras are unique modulo ternary invariants

Theorem

(1) *The clone of each entropic algebra* $\mathcal{A} = \langle A, F, e\rangle$ *with weakly neutral element is determined by its ternary invariants:*

$$\mathrm{Clo}(\mathcal{A}) = \mathrm{Pol}\,\mathrm{Inv}^{(3)} F.$$

(2) *Let* $\mathcal{E}$ *be the class of all entropic algebras with a weakly neutral element. Then the clone of each* $\mathcal{A} \in \mathcal{E}$ *is unique in* $\mathcal{E}$ *modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E},\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B})$$

(3) *Let* $\mathcal{E}_n$ *be the class of all entropic algebras* $\mathcal{A} = \langle A, f, e\rangle$ *with a weakly neutral element and one n-ary fundamental operation. Then each* $\mathcal{A} \in \mathcal{E}_n$ *is unique in* $\mathcal{E}_n$ *modulo ternary invariants:*

$$\mathcal{A}, \mathcal{B} \in \mathcal{E}_n,\ \mathrm{Sub}(\mathcal{A}^3) = \mathrm{Sub}(\mathcal{B}^3) \implies \mathcal{A} = \mathcal{B}.$$

Outline

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \rhd f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \dots, x) = x$*

(3) *f is cyclic commutative, i.e.,*
$f(x_1, x_2, \dots, x_n) = f(x_2, \dots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \ldots, x) = x$*

(3) *f is cyclic commutative, i.e.,*
$f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \ldots, x) = x$*

(3) *f is cyclic commutative, i.e., $f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$*

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \ldots, x) = x$*

(3) *f is cyclic commutative, i.e.,*
$f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \ldots, x) = x$*

(3) *f is cyclic commutative, i.e.,* $f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Algebras $\mathcal{A} = \langle A, f^{\mathcal{A}} \rangle$ (Generalization of semilattices)

Proposition

Let $\mathcal{K}$ be a class of algebras $\mathcal{A} = \langle A, f \rangle$ with one n-ary operation satisfying

(1) *$\mathcal{A}$ is entropic, i.e. f commutes with itself ($f \triangleright f^{\bullet}$),*

(2) *f is idempotent, i.e., $f(x, \dots, x) = x$*

(3) *f is cyclic commutative, i.e.,* $f(x_1, x_2, \dots, x_n) = f(x_2, \dots, x_n, x_1)$

Then the algebras $\mathcal{A} \in \mathcal{K}$ are unique in $\mathcal{K}$ modulo $(n+1)$-ary invariants, i.e.

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Sub}(\mathcal{A}^{n+1}) = \mathrm{Sub}(\mathcal{B}^{n+1}) \implies \mathcal{A} = \mathcal{B}.$$

Examples: $\mathcal{A} = \langle A, \wedge \rangle$ semilattice ($\wedge$ associative, commutative, idempotent)

Crucial Lemma for the proof

Lemma

Let $f, g : A^n \to A$ satisfy:

(1) *f and g commute,* (*(1) implies $g^\bullet \in \mathrm{Inv}^{(n+1)}\{f\}$*)

(2) *f and g are idempotent,*

(3) *f and g are cyclic commutative.*

Then $f = g$.

Proof.

$$f(x_1,\dots,x_n) \overset{(2)}{=} g(f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n))$$
$$\overset{(3)}{=} g(f(x_1,x_2,\dots,x_n),f(x_2,x_3,\dots,x_1),\dots,f(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(1)}{=} f(g(x_1,x_2,\dots,x_n),g(x_2,x_3,\dots,x_1),\dots,g(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(3)}{=} f(g(x_1,\dots,x_n),\dots,g(x_1,\dots,x_n)) \overset{(2)}{=} g(x_1,\dots,x_n)$$

Crucial Lemma for the proof

Lemma

Let $f, g : A^n \to A$ satisfy:

(1) *f and g commute,* ((1) *implies* $g^\bullet \in \mathrm{Inv}^{(n+1)}\{f\}$)

(2) *f and g are idempotent,*

(3) *f and g are cyclic commutative.*

Then $f = g$.

Proof.

$$f(x_1,\dots,x_n) \overset{(2)}{=} g(f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n))$$
$$\overset{(3)}{=} g(f(x_1,x_2,\dots,x_n), f(x_2,x_3,\dots,x_1),\dots,f(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(1)}{=} f(g(x_1,x_2,\dots,x_n), g(x_2,x_3,\dots,x_1),\dots,g(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(3)}{=} f(g(x_1,\dots,x_n),\dots,g(x_1,\dots,x_n)) \overset{(2)}{=} g(x_1,\dots,x_n)$$

Crucial Lemma for the proof

Lemma

Let $f, g : A^n \to A$ satisfy:

(1) *f and g commute,* ((1) *implies* $g^\bullet \in \mathrm{Inv}^{(n+1)}\{f\}$)

(2) *f and g are idempotent,*

(3) *f and g are cyclic commutative.*

Then $f = g$.

Proof.

$$f(x_1,\dots,x_n) \overset{(2)}{=} g(f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n))$$
$$\overset{(3)}{=} g(f(x_1,x_2,\dots,x_n),f(x_2,x_3,\dots,x_1),\dots,f(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(1)}{=} f(g(x_1,x_2,\dots,x_n),g(x_2,x_3,\dots,x_1),\dots,g(x_n,x_1,\dots,x_{n-1}))$$
$$\overset{(3)}{=} f(g(x_1,\dots,x_n),\dots,g(x_1,\dots,x_n)) \overset{(2)}{=} g(x_1,\dots,x_n)$$

Outline

Some open problems

- Which groups are determined by their ternary invariants? $(\mathrm{Clo}(G) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\,G\ ?)$
- Characterize clones with the same local closure.
  Motivation: Proposition. *Let $\mathcal{K}$ be a class of algebras such that*
  $$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B}).$$
  Then the clone of every algebra in $\mathcal{K}$ is unique modulo invariants.
- Find further algebraic properties (P) for operations $f : A^n \to A$ such that the algebras $\langle A, f\rangle$ (or their clone) are unique modulo (n-ary) invariants in the class $\mathcal{K} := \{\langle A, f\rangle \mid f \text{ satisfies } (P)\}$.

Some open problems

- Which groups are determined by their ternary invariants? ($\mathrm{Clo}(G) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\,G$?)
- Characterize clones with the same local closure.
  Motivation: Proposition. *Let $\mathcal{K}$ be a class of algebras such that*
  $$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B}).$$
  Then the clone of every algebra in $\mathcal{K}$ is unique modulo invariants.
- Find further algebraic properties (P) for operations $f : A^n \to A$ such that the algebras $\langle A, f \rangle$ (or their clone) are unique modulo (n-ary) invariants in the class $\mathcal{K} := \{\langle A, f \rangle \mid f \text{ satisfies } (P)\}$.

Some open problems

- Which groups are determined by their ternary invariants? $(\mathrm{Clo}(G) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\,G\ ?)$
- Characterize clones with the same local closure.
  Motivation: Proposition. *Let $\mathcal{K}$ be a class of algebras such that*
  $$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B}).$$
  Then the clone of every algebra in $\mathcal{K}$ is unique modulo invariants.
- Find further algebraic properties (P) for operations $f : A^n \to A$ such that the algebras $\langle A, f \rangle$ (or their clone) are unique modulo (n-ary) invariants in the class $\mathcal{K} := \{\langle A, f \rangle \mid f \text{ satisfies } (P)\}$.

Some open problems

- Which groups are determined by their ternary invariants? $(\mathrm{Clo}(G) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\, G\ ?)$
- Characterize clones with the same local closure.
Motivation: Proposition. *Let $\mathcal{K}$ be a class of algebras such that*

$$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B}).$$

Then the clone of every algebra in $\mathcal{K}$ is unique modulo invariants.

- Find further algebraic properties (P) for operations $f : A^n \to A$ such that the algebras $\langle A, f\rangle$ (or their clone) are unique modulo (n-ary) invariants in the class $\mathcal{K} := \{\langle A, f\rangle \mid f \text{ satisfies } (P)\}$.

Some open problems

- Which groups are determined by their ternary invariants? ($\mathrm{Clo}(G) = \mathrm{Pol}\,\mathrm{Inv}^{(3)}\,G$?)
- Characterize clones with the same local closure.
  Motivation: Proposition. *Let $\mathcal{K}$ be a class of algebras such that*
  $$\forall \mathcal{A}, \mathcal{B} \in \mathcal{K} : \mathrm{Loc}\,\mathrm{Clo}(\mathcal{A}) = \mathrm{Loc}\,\mathrm{Clo}(\mathcal{B}) \implies \mathrm{Clo}(\mathcal{A}) = \mathrm{Clo}(\mathcal{B}).$$
  Then the clone of every algebra in $\mathcal{K}$ is unique modulo invariants.
- Find further algebraic properties (P) for operations $f : A^n \to A$ such that the algebras $\langle A, f \rangle$ (or their clone) are unique modulo (n-ary) invariants in the class $\mathcal{K} := \{\langle A, f \rangle \mid f \text{ satisfies } (P)\}$.

Outline


Thank you for your ATTENTION!

Definition of clone

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all *projections* ($e_i^n(x_1, \ldots, x_n) = x_i$)
- F is *closed under composition* i.e. if $f, g_1, \ldots, g_n \in F$ (f n-ary, g_i m-ary), then

$$f[g_1, \ldots, g_n] \in F$$

$f[g_1, \ldots, g_n](x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$

For arbitrary F, $\langle F \rangle$ (*clone generated by F*) is the least clone containing F.

◂ back

e.g., for a group $G = \langle A, \cdot, ^{-1} \rangle$, the clone $\mathrm{Clo}(G)$ of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1, \ldots, x_n) = x_{i_1}^{s_1} \cdot \ldots \cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $s_j \in \mathbb{N}$, $\in \mathbb{Z}$)

Definition of clone

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all *projections* $(e_i^n(x_1, \ldots, x_n) = x_i)$
- F is *closed under composition* i.e. if $f, g_1, \ldots, g_n \in F$ (f n-ary, g_i m-ary), then

$$f[g_1, \ldots, g_n] \in F$$

$f[g_1, \ldots, g_n](x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$

For arbitrary F, $\langle F \rangle$ (*clone generated by F*) is the least clone containing F.

back

e.g., for a group $G = \langle A, \cdot, ^{-1} \rangle$, the clone $\mathrm{Clo}(G)$ of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1, \ldots, x_n) = x_{i_1}^{s_1} \cdot \ldots \cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $s_j \in \mathbb{N}$, $\in \mathbb{Z}$)

Definition of clone

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all *projections* $(e_i^n(x_1, \dots, x_n) = x_i)$
- F is *closed under composition* i.e. if $f, g_1, \dots, g_n \in F$ (f n-ary, g_i m-ary), then
$$f[g_1, \dots, g_n] \in F$$

$f[g_1, \dots, g_n](x_1, \dots, x_m) := f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$

For arbitrary F, $\langle F \rangle$ (*clone generated by F*) is the least clone containing F.

back

e.g., for a group $G = \langle A, \cdot, ^{-1} \rangle$, the clone $\mathrm{Clo}(G)$ of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1, \dots, x_n) = x_{i_1}^{s_1} \cdot \dots \cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \dots, x_n\}$ and $s_j \in \mathbb{N}$, $\in \mathbb{Z}$)

Definition of clone

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all *projections* $(e_i^n(x_1, \ldots, x_n) = x_i)$
- F is *closed under composition* i.e. if $f, g_1, \ldots, g_n \in F$ (f n-ary, g_i m-ary), then

$$f[g_1, \ldots, g_n] \in F$$

$f[g_1, \ldots, g_n](x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$
For arbitrary F, $\langle F \rangle$ (*clone generated by F*) is the least clone containing F.

back

e.g., for a group $G = \langle A, \cdot, ^{-1} \rangle$, the clone $\mathrm{Clo}(G)$ of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1, \ldots, x_n) = x_{i_1}^{s_1} \cdot \ldots \cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $s_j \in \mathbb{N}, \in \mathbb{Z}$)

Definition of clone

A set F of finitary functions $f : A^n \to A$ (on a base set A) is called *clone*, if

- F contains all *projections* $(e_i^n(x_1, \ldots, x_n) = x_i)$
- F is *closed under composition* i.e. if $f, g_1, \ldots, g_n \in F$ (f n-ary, g_i m-ary), then

$$f[g_1, \ldots, g_n] \in F$$

$f[g_1, \ldots, g_n](x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$

For arbitrary F, $\langle F \rangle$ (*clone generated by F*) is the least clone containing F.

back

e.g., for a group $G = \langle A, \cdot, ^{-1} \rangle$, the clone $\mathrm{Clo}(G)$ of term functions (= clone generated by the operation $x \cdot y$ of multiplication and taking inverse x^{-1}) consists of all functions definable by a semigroup word:

$$f(x_1, \ldots, x_n) = x_{i_1}^{s_1} \cdot \ldots \cdot x_{i_t}^{s_t}$$

(where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $s_j \in \mathbb{N}, \in \mathbb{Z}$)