Normalization of basic algebras

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Abstract

We consider algebras determined by all normal identities of basic algebras. For such algebras, we present a representation based on a q-lattice, i.e. the normalization of a lattice.

Let τ be a similarity type and p, q be n-ary terms of type τ. If either none of them is a variable or both p, q are the same variable, we say that the identity $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ is **normal**.

Let $\mathscr{V}$ be a variety of type τ. Let $\mathrm{Id}(\mathscr{V})$ and $\mathrm{Id}_N(\mathscr{V})$ denote the sets of all identities and of all normal identities, respectively, valid in $\mathscr{V}$. The variety $\mathscr{V}$ is called **normally presentable** if $\mathrm{Id}(\mathscr{V}) = \mathrm{Id}_N(\mathscr{V})$.

If $\mathrm{Id}(\mathscr{V}) \neq \mathrm{Id}_N(\mathscr{V})$ then $\mathscr{V}$ is called **non-normally presentable**. If this is the case then there is a unary term v such that the identity $v(x) = x$ belongs to $\mathrm{Id}(\mathscr{V}) \setminus \mathrm{Id}_N(\mathscr{V})$, see e.g. [Chajda 1995] for details. As usual, for any set Σ of identities of type τ, $\mathrm{Mod}(\Sigma)$ stands for the class of all algebras of type τ that satisfy all identities from Σ.

The following lemma was proved in [Mel'nik 1973].

Lemma

If a non-normally presentable variety $\mathscr{V}$ is given by a system Σ of identities, i.e. $\mathscr{V} = \mathrm{Mod}(\Sigma)$, and $v(x) = x$ belongs to Σ, then there exists a system of normal identities valid in $\mathscr{V}$, $\Sigma_N \subset \mathrm{Id}_N(\mathscr{V})$, such that $\Sigma_N \cup \{v(x) = x\}$ is equivalent to Σ, i.e. $\mathscr{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) = x\})$.

Consequently, $w(x) = x$ is satisfied in $\mathscr{V}$ for another unary term w if and only if the identity $v(x) = w(x)$ belongs to $\mathrm{Id}_N(\mathscr{V})$. So v is determined uniquely up to a normal identity valid in $\mathscr{V}$, and it will be called the **assigned term** of $\mathscr{V}$.

Normalization

A **normalization** of $\mathscr{V}$ (called a **nilpotent shift** of a variety in [Chajda 1995, Chajda–Graczyńska 1999, Mel'nik 1973]) is a variety $N(\mathscr{V}) = \mathrm{Mod}(\mathrm{Id}_N(\mathscr{V}))$. That is, $N(\mathscr{V})$ consists of all τ-algebras which satisfy all normal identities of $\mathscr{V}$. Hence $\mathscr{V}$ is a subvariety of $N(\mathscr{V})$, and $\mathscr{V} = N(\mathscr{V})$ holds if and only if the variety $\mathscr{V}$ is normally presentable.

The next result is taken from [Chajda–Halaš–Kühr–Vanžurová 2005].

Proposition 1.

Let $\mathscr{V}$ be a non-normally presentable variety with an assigned term v. Let $\mathscr{N} = \mathrm{Mod}(\Xi_N)$ be a normally presentable variety with the system of defining identities $\Xi_N \subset \mathrm{Id}_N(\mathscr{V})$. Then $\mathscr{N} = N(\mathscr{V})$ if and only if all defining identities of $\mathscr{V}$ can be proved from the system $\Xi_N \cup \{v(x) = x\}$.

The following proposition was proved by I. Mel'nik:

Proposition 2.

If $\mathscr{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) = x\})$ is a variety of type τ with the set of operation symbols F where $\Sigma_N \subset \mathrm{Id}_N(\mathscr{V})$ then the normalization $N(\mathscr{V})$ is characterized by the identities $\Sigma_N \cup \Sigma_v$ where the set of additional identities is

$$\Sigma_v = \{f(x_1,\ldots,x_n) = v(f(x_1,\ldots,x_n)),$$

$$f(x_1,\ldots,x_j,\ldots,x_n) = f(x_1,\ldots,v(x_j),\ldots,x_n); f \in F, j = 1,\ldots,n\}.$$

Skeleton

Given a non-normally presentable variety $\mathscr{V}$ (of type τ) with the assigned term v, let $A \in N(\mathscr{V})$. By a **skeleton** of A is meant a set $\mathrm{Sk}A = \{a \in A; v^A(a) = a\}$, and its elements are called **skeletal**. Skeletal elements are exactly the results of term operations. In particular, $\mathrm{Sk}A = \{v^A(a); a \in A\}$.

The following lemma was proved in [Chajda 1995].

Lemma

If $A \in \mathscr{V}$ then $\mathrm{Sk}A$ is the maximal subalgebra of A belonging to $N(\mathscr{V})$.

A **quasiorder** on a set A is a reflexive and transitive binary relation $\preceq$ on A, and $(A; \preceq)$ is called a **quasiordered set**.

It is well-known, that lattices have two faces, i.e. they can be viewed as algebras and simultaneously as ordered sets. An analogous situation occurs also for algebras resulting from the normalization of lattices, the so-called q-lattices. A q-lattice can be introduced by identities, but can be characterized as well as a lattice-quasiordered set (with suprema and infima for skeletal elements) endowed with a choice function, [Chajda 1992].

By a **q-lattice** (see [Chajda 1992]) we mean an algebra $A = (A; \vee, \wedge)$ with two binary operations satisfying the following normal identities of lattices:

commutativity: $(\mathrm{C})_\vee : x \vee y = y \vee x$, $(\mathrm{C})_\wedge : x \wedge y = y \wedge x$;

associativity: $(\mathrm{AS})_\vee : (x \vee y) \vee z = x \vee (y \vee z)$, $(\mathrm{AS})_\wedge : (x \wedge y) \wedge z = x \wedge (y \wedge z)$;

weak absorption: $(\mathrm{WAB})_\vee : x \vee (x \wedge y) = x \vee x$, $(\mathrm{WAB})_\wedge : x \wedge (x \vee y) = x \wedge x$;

weak idempotence: $(\mathrm{WI})_\vee : x \vee y = x \vee (y \vee y)$, $(\mathrm{WI})_\wedge : x \wedge y = x \wedge (y \wedge y)$;

equalization: $(\mathrm{EQ}) : x \wedge x = x \vee x$.

A q-lattice A is **bounded** if there exist elements 0 and 1 of A such that $a \wedge 0 = 0$ and $a \vee 1 = 1$ for each $a \in A$.

Evidently, a q-lattice is a lattice if and only if it satisfies the idempotency $x \vee x = x$, i.e. if A is equal to its skeleton.

Proposition 3 [Chajda 1992]

Let $\mathscr{A} = (A; \vee, \wedge)$ be a q-lattice. Define

$$x \preceq y \quad \text{iff} \quad x \vee y = y \vee y \quad (\text{iff} \quad x \wedge y = x \wedge x).$$

Then $\preceq$ is a quasiorder on A such that

(α) for all $x, y \in A$ there exists $z \in A$ such that

- (i) $x, y \preceq z$;
- (ii) if $w \in A$ such that $x, y \preceq w$ then $z \preceq w$,

the element z will be called a **q-supremum** of x, y.

(β) for all $x, y \in A$ there exists $t \in A$ such that

- (i)' $t \preceq x, y$;
- (ii)' if $u \in A$ such that $u \preceq x, y$ then $u \preceq t$,

the element t will be called a **q-infimum** of x, y.

Conversely, let $(A; \preceq)$ be a quasiordered set satisfying the conditions (α) and (β). Define $x \vee y = z$ where z is a q-supremum of x, y and $x \wedge y = t$ where t is a q-infimum of x, y. Then $(A; \vee, \wedge)$ is a q-lattice.

A quasiordered set $(A; \preceq)$ satisfying (α) where $x \vee y$ denote q-supremum of x, y is called a **join-q-semilattice.**

A **basic algebra** (see [Chajda–Halaš–Kühr 2007]) is an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$;

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

Clearly, also the (normal) identities $\neg\neg x = x \oplus 0$ and $\neg\neg\neg x = \neg x$ hold in every basic algebra.

Let us note that basic algebras serve as a tool for some investigations of nonclassical logics (including MV-algebras, orthomodular lattices and their generalizations).

The basic algebras form a variety **BA** which is not normally presentable, with $v(x) = x \oplus 0$ as the assigned term (or equivalently, $v(x) = \neg\neg x$). According to Proposition 2, the normalization $N(\mathbf{BA})$ has a basis consisting of the following normal identities:

(N1) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x;$

(N2) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0;$

(N3) $0 \oplus 0 = 0;$

(N4) $\neg\neg x = x \oplus 0;$

(N5) $x \oplus y = (x \oplus 0) \oplus y;$

(N6) $x \oplus y = x \oplus (y \oplus 0);$

(N7) $x \oplus \neg 0 = \neg 0;$

(N8) $\neg 0 \oplus x = \neg 0;$

(N9) $(x \oplus y) \oplus 0 = x \oplus y;$

(N10) $\neg\neg\neg x = \neg x;$

(N11) $\neg(x \oplus 0) = \neg x;$

(N12) $\neg x \oplus 0 = \neg x;$

Thus $N(\mathbf{BA}) = \mathrm{Mod}(\mathrm{Id}_N(\mathbf{BA})) = \mathrm{Mod}(\{(\mathrm{N1}) - (\mathrm{N12})\})$.
One can show that this axiom system can be reduced.

..., so $N(\mathbf{BA}) = \mathrm{Mod}(\{(\mathrm{N1}) - (\mathrm{N10})\})$. Since $v(x) = x \oplus 0$, the skeleton of a basic algebra $\mathscr{M} = (M; \oplus, \neg, 0)$ is $\mathrm{Sk}M = \{a \oplus 0; a \in M\}$.

It is known (see e.g. [Chajda–Halaš–Kühr 2007]) that basic algebras form bounded lattices with respect to the natural order defined by $x \leq y$ if and only if $\neg x \oplus y = \neg 0$ where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. An analogous statement can be proved for their normalizations:

Theorem 1

Let $\mathscr{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$. Define $x \preceq y$ if and only if $\neg x \oplus y = \neg 0$. Then $(A; \preceq)$ is a bounded q-lattice with $0 \preceq x \preceq \neg 0$ for each $x \in A$ and $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$.

Example

Let us consider the algebra $\mathscr{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$, where $A = \{0, 0', a, a', b, 1\}$, and whose operations $\oplus$ and $\neg$ are given by the following tables

$\oplus$	0	$0'$	a	a'	b	1
0	0	0	a	a	b	1
$0'$	0	0	a	a	b	1
a	a	a	1	1	b	1
a'	a	a	1	1	b	1
b	b	b	a	a	1	1
1	1	1	1	1	1	1

x	0	$0'$	a	a'	b	1
$\neg x$	1	1	a	a	b	0

Note that e.g. $a' \oplus 0 \neq a'$, $\neg\neg a' \neq a'$.

Example

By Theorem 1, we can assign to $\mathscr{A}$ a bounded q-lattice $\mathscr{Q} = (A; \vee, \wedge)$, where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for all $x, y \in A$. The tables for operations $\vee$ and $\wedge$ in $\mathscr{Q}$ are as follows

$\vee$	0	$0'$	a	a'	b	1
0	0	0	a	a	b	1
$0'$	0	0	a	a	b	1
a	a	a	a	a	1	1
a'	a	a	a	a	1	1
b	b	b	1	1	b	1
1	1	1	1	1	1	1

$\wedge$	0	$0'$	a	a'	b	1
0	0	0	0	0	0	0
$0'$	0	0	0	0	0	0
a	0	0	a	a	0	a
a'	0	0	a	a	0	a
b	0	0	0	0	b	b
1	0	0	a	a	b	1

Example

One can easily draw the diagram of $\mathscr{Q}$ in Fig. 1.

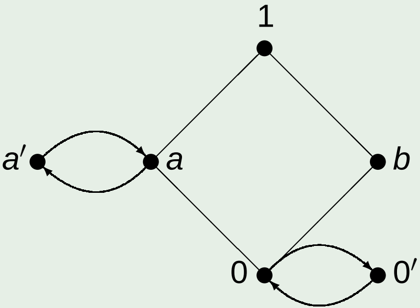

Fig. 1

Remark that $1 = \neg 0$ is the greatest element of $\mathscr{Q}$, but 0 is not the least element of $\mathscr{Q}$, since $0 \preceq 0'$ and also $0' \preceq 0$.

Example

The Hasse diagram of the skeleton of $\mathscr{Q}$ is depicted in Fig. 2; of course it is a basic algebra.

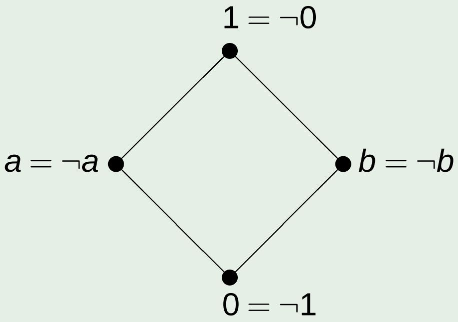

Fig. 2

As usual, under an **involution** on a set A we mean a map ${}^p : A \to A$ such that $a^{pp} = a$ for all $a \in A$.

Given a quasiordered set $(A; \preceq)$, a map ${}^p : A \to A$ is called **antitone** if the implication $x \preceq y \Rightarrow y^p \preceq x^p$ holds.

Let $\mathscr{L} = (L; \vee, \wedge, 1)$ be a q-lattice with the greatest idempotent 1 (i.e. $1 = 1 \vee 1$), and let $\preceq$ denote the induced quasiorder on L. Remember that the skeleton $\mathrm{Sk}\mathscr{L} = \{x \in L; x \vee x = x\}$ is a lattice. By an **interval** in $\mathscr{L}$ we understand here the set $[a, b] = \{x \in L; a \preceq x \preceq b\}$, and under an **interval in the skeleton** the intersection $\mathrm{Sk}[a, b] = \mathrm{Sk}L \cap [a, b]$ provided $a, b \in \mathrm{Sk}L$. For example, $[0, a] = \{0, 0', a, a'\}$ and $\mathrm{Sk}[0, a] = \{0, a\}$ for the q-lattice of Example (see Fig. 1).

q-lattices with sectionally antitone mappings

Remark

For any $p \in L$, let an antitone involution ${}^p : x \mapsto x^p$, $x \in \mathrm{Sk}L$, be given on the interval $\mathrm{Sk}[p \vee p, 1]$. The mapping ^{p} with $p \in L$ can be extended to a mapping on the whole interval $[p, 1]$ in a natural way as follows. For $x \in [p, 1]$ we define $x^p := (x \vee x)^{p \vee p}$. Note that in general, $x \mapsto x^p$ is not an involution on $[p, 1]$ but only on $\mathrm{Sk}[p \vee p, 1]$. Indeed, $x^{pp} = ((x \vee x)^{p \vee p} \vee (x \vee x)^{p \vee p})^{p \vee p} = ((x \vee x)^{p \vee p})^{p \vee p} = x \vee x \in \mathrm{Sk}L$, however $x^{pp} \neq x$ for $x \notin \mathrm{Sk}L$.

Lemma

Let $\mathscr{L} = (L; \vee, \wedge, 1)$ *be a q-lattice with* $1 = 1 \vee 1$*. For any* $p \in L$*, let an antitone mapping* $^p : x \mapsto x^p$*, be given on the interval* $[p \vee p, 1]$ *such that its restriction to* $\mathrm{Sk}[p \vee p, 1]$ *is an involution. For* $x, y \in L$*, let us introduce a binary operation* $x \circ y := (x \vee y)^{y \vee y}$*. Then the following identities hold:*

(1) $x \circ x = 1$, $x \circ 1 = 1$;

(2) $1 \circ (x \circ y) = x \circ y$;

(3) $(x \circ y) \circ y = (y \circ x) \circ x$ *(quasi-commutativity);*

(4) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1$;

(5) $x \circ ((x \circ y) \circ y) = 1$.

Moreover,

(6) *if* $x \vee y \vee z = z \vee z$ *then* $((x \circ y) \circ y) \circ z = 1$;

(7) *if* $x \vee y = y \vee y$ *then* $(y \circ z) \circ (x \circ z) = 1$.

Lemma

Let $(A;\circ,1)$ be an algebra of type $(2,0)$ satisfying the identities (1), (2) and (4). Then the relation $\preceq$ introduced by

$$x \preceq y \qquad \textit{if and only if} \qquad x \circ y = 1$$

is a quasiorder on A and for all $x \in A$, we have $x \preceq 1$. Moreover, $x \circ y = 1$ if and only if $x \vee y = y \vee y$.

The quasiorder $\preceq$ given by $x \preceq y \Leftrightarrow x \circ y = 1$ will be called the **induced quasiorder** of $(A;\circ,1)$.

Theorem 2

Let $\mathscr{A} = (A; \circ, 1)$ be an algebra satisfying (1)–(7). Then $(A; \preceq)$ is a join-q-semilattice in which $x \vee y = (x \circ y) \circ y$ for all $x, y \in A$. For each $p \in A$, the interval $[p \vee p, 1]$ is a q-lattice with an antitone mapping $a \mapsto a^p = a \circ p$, $a \in [p \vee p, 1]$.

Theorem 3

Let $\mathscr{A} = (A; \oplus, \neg, 0) \in N(\mathbf{BA})$. Define $x \circ y := \neg x \oplus y$ and $1 = \neg 0$. Further, let $\vee$, $\wedge$ are defined as in Theorem 1, i.e. $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $\mathscr{L}(A) = (A; \vee, \wedge, \circ, 1, 0)$ is a bounded q-lattice with sectionally antitone mappings such that their restrictions to $\mathrm{Sk}[p \vee p, 1]$ are involutions where for each $p \in A$ and $x \in [p \vee p, 1]$ we define $x^p = x \circ p$.

Theorem 4

Let $\mathscr{L} = (L; \vee, \wedge, \circ, 0, 1)$ be a bounded q-lattice with sectionally antitone mappings such that their restrictions to $\mathrm{Sk}[p \vee p, 1]$ are involutions. Define $\neg x := x \circ 0$ and $x \oplus y := (x \circ 0) \circ y$. Then $\mathscr{A}(\mathscr{L}) = (L; \oplus, \neg, 0) \in N(\mathbf{BA})$.

Theorem 5

Let $\mathscr{A} = (A; \circ, 1)$ be an algebra satisfying (1)–(7) where $x \vee y = (x \circ y) \circ y$. Let $p \in A$ with $1 \circ p = p$ and define $\neg_p x := x \circ p$, $x \oplus_p y := (x \circ p) \circ y$. Then the algebra $([p, 1]; \oplus_p, \neg_p, p)$ belongs to $N(\mathbf{BA})$.

Thank you for your attention.

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