Many independent equality constraints

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Outline

1. Formulas and primitive positive definitions
2. PP-closed relational structures
3. Local clones


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Let Σ be a set of equality constraints.
Σ is called an *equality constraint language*.

Primitive positive definitions and the problem

Definition

ϕ is *pp-definable* (*primitively positively definable*) from $\Sigma \leftrightarrow$ ϕ is logically equivalent to a formula of the form $\exists x_{j_1} \ldots \exists x_{j_n}\, \phi_1 \wedge \ldots \wedge \phi_m$, with $\phi_i \in \Sigma \cup \{x = y\}$.

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Problem (Formulas)

Are there uncountably many pp-inequivalent equality constraint languages?

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- M. Junker and M. Ziegler: There are 116 reducts of $(\mathbb{Q}, <, 0)$ up to f.o.-interdefinability.

Reducts up to pp-interdefinability

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Problem (Reducts)

Is the number of reducts of Γ uncountable (up to pp-interdefinability)?

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Reducts of $(X, =)$ as templates of CSPs.

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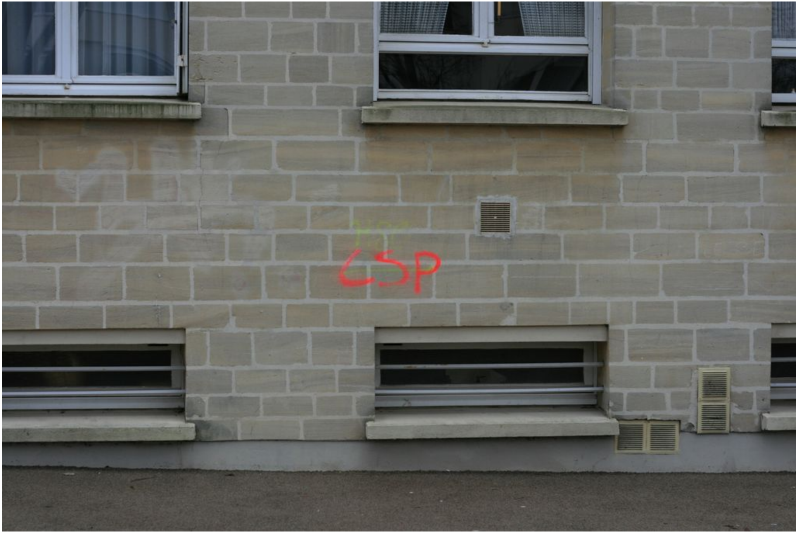
(Bodirsky – Chen – Kara)

CSP fanatics in Caen



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$\mathrm{Pol}(\Gamma) := \{f \in \mathcal{O} : f$ preserves all relations of $\Gamma\}$.
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Theorem (Bodirsky-Nešetřil)

Let Γ *be* ω*-categorical. Then* $\mathrm{Inv}\,\mathrm{Pol}(\Gamma) = pp(\Gamma)$.

Local clones

Definition

$\mathcal{C} \subseteq \mathcal{O}$ is a *clone* $\leftrightarrow$

- $\mathcal{C}$ is closed under composition
  I.e. $f(g_1, \ldots, g_n) \in \mathcal{C}$ for all $f, g_1, \ldots, g_n \in \mathcal{C}$, and
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The local clones are exactly the Pol Inv-closed subsets of $\mathcal{O}$.

Local clones and pp-closed reducts

Observations

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More observations

Inv (or Pol) is an antiisomorphism between

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Answer to: Formulas (strong)

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Answer (Formulas strong)

Yes.

The formulas

For all $n \geq 3$, write

$$\delta_n := x_1 \neq y_1 \vee \ldots \vee x_n \neq y_n.$$

For all $A \subseteq \{1, \ldots, n\}$ with $1 < |A| < n$, writing $A = \{j_1, \ldots, j_r\}$ with $j_1 < j_2 < \ldots < j_r$, we set

$$\kappa_A := y_{j_1} \neq x_{j_2} \vee y_{j_2} \neq x_{j_3} \vee \ldots \vee y_{j_r} \neq x_{j_1}.$$

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Theorem (M. Bodirsky, H. Chen, MP 2008)

$\{\phi_n : n \geq 3\}$ *is independent.*

Reference

M. Bodirsky, H. Chen, M. Pinsker,
The reducts of equality up to primitive positive interdefinability,
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