

# Nice semigroup varieties with large free spectra

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5th September 2008

(Higman (1967), Neumann (1963)) If  $\mathcal{V}$  is a locally finite group variety, then the size of the free group generated by  $n$  elements in the variety is

- exponential in  $n$ , if  $\mathcal{V}$  is nilpotent,
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# Groups

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- exponential in  $n$ , if  $\mathcal{V}$  is nilpotent,
- at least double exponential, if  $\mathcal{V}$  is not nilpotent.

where

- $\mathcal{V}$  locally finite : every finitely generated algebra is finite in  $\mathcal{V}$
- $\mathcal{V}$  nilpotent: every group in  $\mathcal{V}$  is nilpotent

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- Abelian group of exponent  $d$ :

terms:  $\prod x_i^{d_i}, 0 \leq d_i \leq d$

$$|\mathbf{F}_{\mathcal{V}}(n)| = d^n$$



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terms:  $\prod x_i^{d_i}$ ,  $0 \leq d_i \leq d$   
 $|\mathbf{F}_{\mathcal{V}}(n)| = d^n$
- Boolean algebra:  $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$

# Trivial estimate

**A** algebra, if  $|\mathbf{A}| = k > 1$ , then

$$n \leq |\mathbf{F}_{\mathcal{V}(\mathcal{A})}(n)| \leq k^{k^n}$$

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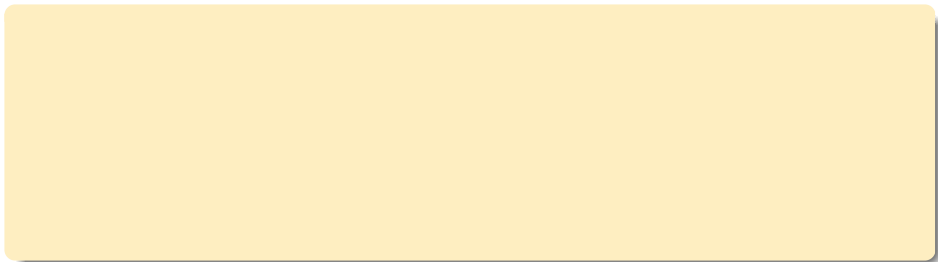
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$n$ : the number of projections

$k^{k^n}$ : the number of functions

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- there are free spectra such that for an integer  $k$   $\log \log \dots \log |\mathbf{F}_{\mathcal{V}}(n)|$  is bounded by a polynomial (log is iterated  $k$ -times)
- no other free spectra is known



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- semilattices (union of one element groups)
- idempotent semigroups (bands)

Interesting fact

the completely regular semigroups form a variety

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## Note

A locally finite semigroup variety is a CRS variety if and only if it satisfies the identity

$$x^r = x, \text{ for some } r.$$



## $\rho_n$ sequence

Let  $t = t(x_1, \dots, x_n)$  be an  $n$ -ary term. A term operation  $t^{\mathbf{A}}$  is said to be **essentially  $n$ -ary**, if it depends on all of its variables, i.e. if for all  $1 \leq i \leq n$  there exist  $a_1, \dots, a_{i-1}, a, b, a_{i+1}, \dots, a_n \in A$  such that

$$t(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) \neq t(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

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$\rho_n(\mathbf{A})$ : the number of essentially  $n$ -ary terms over  $\mathbf{A}$

## $p_n$ sequence

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$p_n(\mathbf{A})$ : the number of essentially  $n$ -ary terms over  $\mathbf{A}$

$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{k=0}^n \binom{n}{k} p_k(\mathbf{A})$$

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- the semigroup variety defined by  $x^r = x$  is locally finite  $\iff$  the group variety defined by  $x^{r-1} = 1$  is locally finite.
- a recurrence formula for  $p_n$



# The recurrence formula

Green, Rees

$$p_n = n^2 p_{n-1}^2 |G_m| \text{ for some } m, \text{ where}$$

$G_m$  is the  $m$ -generated free group in the group variety defined by  $x^{r-1} = 1$ .

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- $|G_m| = 1$

- $p_n \sim \frac{1}{n^2} K^{2^{n+1}}$ , where  $K = \sqrt{2\sqrt{3\sqrt{4\sqrt{5\dots}}}} \sim 1,661687$

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- $n = 3 \rightarrow m = 374$
- $n = 4 \rightarrow m = ??$

$m=?$

$x^r = x$  (Kadourek, Polák (1988-1990))

$m$ : complicated recursive description of the generators of  $G_m$  depending on the structure of  $F_{\mathcal{V}}(n-1)$ .

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$x^r = x$

$$m = \sum_1^{n-1} \binom{n}{k} p_k(\mathcal{V}_r) \left( 1 - \frac{2}{r-1} \sum_{t=1}^k \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r)} + \frac{1}{(r-1)^2} \sum_{t+h \leq k} \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r) \binom{n}{h} h p_{h-1}(\mathcal{V}_r)} \right) + np_{n-1}(\mathcal{V}_r) \left( 1 - \frac{2}{r-1} \sum_{t=1}^n \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r)} + \right)$$

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where there are  $n$ -many 2-s.

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  - the idempotens of  $S$  form a subsemigroup
- double exponential, otherwise.