Nice semigroup varieties with large free spectra

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Groups

(Higman (1967), Neumann (1963)) If $\mathcal{V}$ is a locally finite group variety, then the size of the free group generated by n elements in the varitey is

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where

- $\mathcal{V}$ locally finite : every finitely generated algebra is finite in $\mathcal{V}$
- $\mathcal{V}$ nilpotent: every group in $\mathcal{V}$ is nilpotent

Free spectrum

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  $|\mathbf{F}_{\mathcal{V}}(n)| = d^n$
- Boolean algebra: $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$

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n: the number of projections
k^{k^n}: the number of functions

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- there are free spectra such that for an integer k $\log \log \ldots \log |\mathbf{F}_{\mathcal{V}}(n)|$ is bounded by a polynomial (log is iterated k-times)
- no other free spectra is known

Completely regular semigroups (CRS)

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- idempotent semigroups (bands)

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Note

A locally finite semigroup variety is a CRS variety if and only if it satisfies the identity

$$x^r = x, \text{ for some } r.$$

p_n sequence

Let $t = t(x_1, \ldots, x_n)$ be an n-ary term. A term operation $t^{\mathbf{A}}$ is said to be essentially n-ary, if it depends on all of its variables, i.e. if for all $1 \leq i \leq n$ there exist $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n \in A$ such that

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \neq t(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

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$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$$

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- a recurrence formula for p_n

The recurrence formula

Green, Rees

$$p_n = n^2 p_{n-1}^2 |G_m| \text{ for some } m \text{, where}$$

G_m is the m-generated free group in the group variety defined by $x^{r-1} = 1$.

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- $|G_m| = 1$
- $p_n \sim \frac{1}{n^2} K^{2^{n+1}}$, where $K = \sqrt{2\sqrt{3\sqrt{4\sqrt{5\ldots}}}} \sim 1,661687$

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- $n = 4 \rightarrow m = ??$

m=?

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$x^r = x$

$$m = \sum_{1}^{n-1} \binom{n}{k} p_k(\mathcal{V}_r) \left(1 - \frac{2}{r-1} \sum_{t=1}^{k} \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r)} + \right.$$

$$\left. + \frac{1}{(r-1)^2} \sum_{t+h \leq k} \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r) \binom{n}{h} h p_{h-1}(\mathcal{V}_r)} \right) +$$

$$+ n p_{n-1}(\mathcal{V}_r) \left(1 - \frac{2}{r-1} \sum_{t=1}^{n} \frac{1}{\binom{n}{t} t p_{t-1}(\mathcal{V}_r)} + \right.$$

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Finitely generated CRS varieties

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- exponential if and only if
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- double exponential, otherwise.