Universal Algebra and Computational Complexity
Lecture 3

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Třešť, September 2008

Summary of Lecture 2

Recall from Tuesday:

L	$\subseteq$	NL	$\subseteq$	P	$\subseteq$	NP	$\subseteq$	$PSPACE$	$\subseteq$	$EXPTIME$ $\cdots$
$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$
$FVAL$, $2COL$		$PATH$, $2SAT$		$CVAL$, $HORN$-$3SAT$		SAT, $3SAT$, $3COL$, $4COL$, etc. $HAMPATH$		1-CLO		CLO

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Today:

- Some decision problems involving finite algebras
- How hard are they?

Encoding finite algebras: size matters

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For each fundamental operation f: If $\text{arity}(f) = r$, then f is given by its *table*, having ...

- n^r entries;
- each entry requires $\log n$ bits.

The tables (as bit-streams) must be separated from each other by $\#$'s.

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Hence the size of **A** is

$$||\mathbf{A}|| = \sum_{\textit{fund } f} \left(n^{\mathrm{arity}(f)} \log n + 1 \right).$$

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$$\begin{aligned} R &= \text{maximum arity of the fundamental operations (assume } > 0) \\ T &= \text{number of fundamental operations (assume } > 0). \end{aligned}$$

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In particular, if we restrict our attention to algebras with some **fixed** number T of operations, then

$$||\mathbf{A}|| \;\sim\; n^R \log n.$$

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INPUT: A finite algebra $\mathbf{A}$ and two terms $s(\vec{x}), t(\vec{x})$.

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How hard are these problems?

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Ideally, we want to find an $X \in \{L, NL, P, NP, \ldots\}$ which is both an upper and a lower bound to the complexity of D ...

- ...i.e., such that D is X-complete.

An easy problem: Subalgebra Membership (*SUB-MEM*)

Subalgebra Membership Problem (*SUB-MEM*)

INPUT:

- An algebra $\mathbf{A}$.
- A set $S \subseteq A$.
- An element $b \in A$.

QUESTION: Is $b \in \mathrm{Sg}^{\mathbf{A}}(S)$?

How hard is *SUB-MEM*?

An obvious upper bound for *SUB-MEM*

Algorithm:

INPUT: $\mathbf{A}, S, b$.

$S_0 := S$

For $i = 1, \ldots, n$ ($:= |A|$)

$S_i := S_{i-1}$

For each operation f (of arity r)

For each $(a_1, \ldots, a_r) \in (S_{i-1})^r$

$c := f(a_1, \ldots, a_r)$

$S_i := S_i \cup \{c\}$.

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OUTPUT: whether $b \in S_n$.

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Heuristics:
$n\left(\sum_f n^{\mathrm{ar}(f)}\right) \leq$ $n\|\|\mathbf{A}\|\|$ steps

The Complexity of *SUB-MEM*

So $\mathit{SUB\text{-}MEM} \in \mathit{TIME}(N^2)$, or maybe $\mathit{TIME}(N^{4+\epsilon})$, or surely in $\mathit{TIME}(N^{55})$, and so we get the "obvious" upper bound:

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Theorem (N. Jones & W. Laaser, '77)

Yes.
In other words, SUB-MEM is P-complete.

A variation: 1-*SUB-MEM*

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This is the restriction of *SUB-MEM* to unary algebras (all fundamental operations are unary). I.e.,

INPUT: A *unary* algebra $\mathbf{A}$, a set $S \subseteq A$, and $b \in A$.

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Here is a nondeterministic log-space algorithm showing 1-*SUB-MEM* $\in$ *NL*:

NALGORITHM: guess a sequence $c_0, c_1, \ldots, c_k$ such that

- $c_0 \in S$
- For each $i < k$, $c_{i+1} = f_j(c_i)$ for some fundamental operation f_j
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Theorem (N. Jones, Y. Lien & W. Laaser, '76)

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2. Given $\mathbf{A}$ and $S \subseteq A$, determine whether S is a subalgebra of $\mathbf{A}$.

$$S \in \mathrm{Sub}(\mathbf{A}) \iff \forall a \in A (a \in \mathrm{Sg}^{\mathbf{A}}(S) \to a \in S).$$

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5. Given $\mathbf{A}$, determine whether $\mathbf{A}$ is simple.

$$\mathbf{A} \text{ simple} \quad \Leftrightarrow \quad \forall a,b,c,d[c \neq d \ \to\ (a,b) \in \mathrm{Cg}^{\mathbf{A}}(c,d)].$$

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6. Given $\mathbf{A}$, determine whether $\mathbf{A}$ is abelian.

$$\mathbf{A} \text{ abelian} \quad \Leftrightarrow \quad \forall a,c,d[c \neq d \ \to\ ((a,a),(c,d)) \notin \mathrm{Cg}^{\mathbf{A}^2}(0_A)].$$

Clone Membership Problem (*CLO*)

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The running time is bounded by a polynomial in $||\mathbf{A}^{(A^k)}||$.

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Can show

$$\log ||\mathbf{A}^{(A^k)}|| \leq n^k ||\mathbf{A}|| \leq (||g|| + ||\mathbf{A}||)^2.$$

Hence the running time is bounded by the exponential of a polynomial in the size of the input $(\mathbf{A}, g)$. I.e., $CLO \in EXPTIME$.

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By reducing a known *EXPTIME*-complete problem to *CLO*, Friedman and Bergman *et al* showed:

Theorem

CLO is EXPTIME-complete.

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QUESTION: Is **A** primal?

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The obvious algorithm is actually a reduction to *CLO*.

For a finite set A, let g_A be your favorite binary Sheffer operation on A.

Define $f : PRIMAL_{inp} \rightarrow CLO_{inp}$ by

$$f : \mathbf{A} \mapsto (\mathbf{A}, g_A).$$

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Since

$$\mathbf{A} \text{ is primal} \quad \Leftrightarrow \quad g_A \in \mathrm{Clo}\,\mathbf{A},$$

we have $PRIMAL \leq_f CLO$. Clearly f is P-computable, so

$$PRIMAL \leq_P CLO$$

which gives the obvious upper bound

$$PRIMAL \in EXPTIME.$$

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But testing primality of algebras is special. Maybe there is a better, “nonobvious” algorithm?

(E.g., using Rosenberg’s classification?)

Open Problem 1.

Determine the complexity of *PRIMAL*.

- Is it in *PSPACE*? ($=$ *NPSPACE*)
- Is it *EXPTIME*-complete? ($\Leftrightarrow$ $CLO \leq_P PRIMAL$)

MALTSEV

INPUT: a finite algebra **A**.

QUESTION: Does **A** have a Maltsev term?

The obvious upper bound is *NEXPTIME*, since *MALTSEV* is a projection of

$$\{\,(\mathbf{A},p)\;:\;\underbrace{p\in\mathrm{Clo}\,\mathbf{A}}_{EXPTIME}\ \text{and}\ \underbrace{p \text{ is a Maltsev operation}}_{P}\,\},$$

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a problem in *EXPTIME*.

But a slightly less obvious algorithm puts *MALTSEV* in *EXPTIME*. Use the fact that if x, y name the two projections $A^2 \to A$, then $\mathbf{A}$ has a Maltsev term iff

$$(y,x) \in \operatorname{Sg}^{\mathbf{A}^{(A^2)}}((x,x),(x,y),(y,y))$$

(which is decidable in *EXPTIME*).

Similar characterizations give *EXPTIME* as an upper bound to the following:

Some problems in *EXPTIME*

Given **A**:

1. Does **A** have a majority term?
2. Does **A** have a semilattice term?
3. Does **A** have Jónsson terms?
4. Does **A** have Gumm terms?
5. Does **A** have terms equivalent to **V**(**A**) being congruence meet-semidistributive?
6. Etc. etc.

Are these problems easier than *EXPTIME*, or *EXPTIME*-complete?

For some of these problems we have an answer:

Theorem (R. Freese, M. Valeriote, '0?)

The following problems are all EXPTIME-complete:
Given $\mathbf{A}$,

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4. *Does* **A** *have a semilattice term?*
5. *Does* **A** *have any nontrivial idempotent term?*
   - *idempotent* means "satisfies $f(x, x, \ldots, x) \approx x$."
   - *nontrivial* means "other than x."

Freese & Valeriote's theorem

Proof.

Freese and Valeriote give a construction which, given an input $\Gamma = (\mathbf{A}, g)$ to *CLO*, produces an algebra $\mathbf{B}_\Gamma$ such that:

- $g \in \mathrm{Clo}\,\mathbf{A} \;\Rightarrow\;$ there is a flat semilattice order on B_Γ such that $(x \wedge y) \vee (x \wedge z)$ is a term operation of $\mathbf{B}_\Gamma$.
- $g \notin \mathrm{Clo}\,\mathbf{A} \;\Rightarrow\;$ $\mathbf{B}_\Gamma$ has no nontrivial idempotent term operations.

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Moreover, the function $f : \Gamma \mapsto \mathbf{B}_\Gamma$ is easily computed (in $\mathbf{P}$).

Hence f is simultaneously a P-reduction of *CLO* to all the problems in the statement of the theorem. □

Open Problem 2.

Are the following easier than *EXPTIME*, or *EXPTIME*-complete?

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Theorem

A *is primal iff:*

- **A** *has no proper subalgebras,*
- **A** *is simple,*
- **A** *is rigid,*
- **A** *is not abelian, and*
- **A** *is Maltsev.*

} *in P* (applies to the first four conditions)

Surprisingly, the previous problems become significantly easier when restricted to *idempotent* algebras.

Theorem (Freese & Valeriote, '0?)

The following problems for ***idempotent*** *algebras are in* **P**:

1. **A** *has a majority term.*
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Proof.

Fiendishly nonobvious algorithms using tame congruence theory. □

Variety Membership Problem (*VAR-MEM*)

INPUT: two finite algebras $\mathbf{A}, \mathbf{B}$ in the same signature.

QUESTION: Is $\mathbf{A} \in \mathbf{V}(\mathbf{B})$?

The obvious algorithm (J. Kalicki, '52): determine whether the identity map on A extends to a homomorphism $\mathbf{F}_{\mathbf{V}(\mathbf{B})}(A) \to \mathbf{A}$.

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$$2\text{-}EXPTIME \stackrel{def}{=} \bigcup_{k=1}^{\infty} TIME(2^{(2^{O(N^k)})})$$

$$\cdots NEXPTIME \subseteq EXPSPACE \subseteq 2\text{-}EXPTIME \subseteq N(2\text{-}EXPTIME) \cdots$$

What is the "real" complexity of *VAR-MEM*?

Theorem (Z. Székely, thesis '00)

VAR-MEM is NP-hard (i.e., 3SAT $\leq_P$ *VAR-MEM).*

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Theorem (M. Kozik, '0?)

VAR-MEM is 2-EXPTIME-hard and therefore 2-EXPTIME-complete. Moreover, there exists a specific finite algebra $\mathbf{B}$ *such that the subproblem:*

> *INPUT: a finite algebra* $\mathbf{A}$ *in the same signature as* $\mathbf{B}$.
>
> *QUESTION: Is* $\mathbf{A} \in \mathbf{V}(\mathbf{B})$

is 2-EXPTIME-complete.

The Equivalence of Terms problem (*EQUIV-TERM*)

INPUT:

- A finite algebra $\mathbf{A}$.
- Two terms $s(\vec{x}), t(\vec{x})$ in the signature of $\mathbf{A}$.

QUESTION: Is $s(\vec{x}) \approx t(\vec{x})$ identically true in $\mathbf{A}$?

It is convenient to name the *negation* of this problem:

The Inequivalence of Terms problem (*INEQUIV-TERM*)

INPUT: (same)

QUESTION: Does $s(\vec{x}) \neq t(\vec{x})$ have a solution in $\mathbf{A}$?

How hard are these problems?

Obviously *INEQUIV-TERM* is in *NP*. (Any solution $\vec{x}$ to $s(\vec{x}) \neq t(\vec{x})$ serves as a certificate.)

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Definition

- Co-*NP* is the class of problems D whose negation $\neg D$ is in *NP*.
- A problem D is co-*NP*-complete if its negation $\neg D$ is *NP*-complete, or equivalently, if D is in the top $\equiv_P$-class of co-*NP*.

Done. End of story. Boring.

But WAIT!!!! There's more!!!!

For each fixed finite algebra **A** we can pose the subproblem for **A**:

EQUIV-TERM(**A**)

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Problem: for which finite algebras $\mathbf{A}$ is $EQUIV\text{-}TERM(\mathbf{A})$ NP-complete? For which $\mathbf{A}$ is it in P?

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Theorem (H. Hunt & R. Stearns, '90; S. Burris & J. Lawrence, '93)

Let **R** *be a finite ring.*

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And many partial results for **semigroups** due to e.g. Kisielewicz, Klíma, Pleshcheva, Popov, Seif, Szabó, Tesson, Therien, Vértesi, and Volkov.

An outrageous scandal

An outrageous scandal

Theorem (G. Horváth & C. Szabó)

Consider the group $\mathbf{A}_4$.

- *EQUIV-TERM*($\mathbf{A}_4$) *is in P.*
- *Yet there is an algebra* $\mathbf{A}$ *with the same clone as* $\mathbf{A}_4$ *such that EQUIV-TERM*($\mathbf{A}$) *is co-NP-complete.*

This is either wonderful or scandalous.

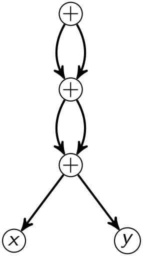
In my opinion, this is evidence that *EQUIV-TERM* is the wrong problem.

Definition

A circuit (in a given signature for algebras) is an object, similar to a term, except that repeated subterms need be written only once.

Example: Let $t = ((x + y) + (x + y)) + ((x + y) + (x + y))$.

A circuit for t:



Straight-line program:

$$\begin{aligned} v_1 &= x + y \\ v_2 &= v_1 + v_1 \\ t &= v_2 + v_2. \end{aligned}$$

Note that circuits may be significantly shorter than the terms they represent.

Equivalence of Terms Problem (correct version)

Fix a finite algebra **A**.

The Equivalence of Circuits problem (*EQUIV-CIRC*(A**))**

INPUT: two circuits $s(\vec{x}), t(\vec{x})$ in the signature of **A**.

QUESTION: is $s(\vec{x}) \approx t(\vec{x})$ identically true in **A**?

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Open Problem 3.

For which finite algebras **A** is *EQUIV-CIRC*(**A**) *NP*-complete? For which **A** is it in *P*?

Two problems for relational structures

Relational Clone Membership (*RCLO*)

INPUT:

- A finite relational structure $\mathbf{M}$.
- A finitary relation $R \subseteq M^k$.

QUESTION: Is $R \in \operatorname{Inv}\operatorname{Pol}(\mathbf{M})$?

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Open Problem 4.

Is *RCLO* in *EXPTIME*? Is it *NEXPTIME*-complete?

Fix a finite relational structure **B**.

Consider the following problem associated to **B**:

A problem

INPUT: a finite structure **A** in the same signature as **B**.

QUESTION: Is there a homomorphism $h : \mathbf{A} \to \mathbf{B}$?

This problem is called $CSP(\mathbf{B})$.

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If $\mathbf{K}_3$ is the triangle graph, then $CSP(\mathbf{K}_3) = 3COL$, so is NP-complete in this case. If $\mathbf{G}$ is a bipartite graph, then then $CSP(\mathbf{G}) \in P$.

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CSP Classification Problem

For which finite relational structures **B** is $CSP(\mathbf{B})$ in P? For which is it NP-complete?