Universal Algebra and Computational Complexity
Lecture 2

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Summary of Lecture 1

Recall from yesterday:

$$\begin{array}{ccccccc} L & \subseteq & P & \subseteq & PSPACE & \subseteq & EXPTIME \\ \Cup & & \Cup & & \Cup & & \Cup \\ FVAL & & PATH & & 3COL & & CLO \end{array}$$

Topics for today:

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Topics for today:

- "Nondeterministic" complexity classes
- Reductions
- Complete problems

"Nondeterministic polynomial time": an example

Recall

Graph 3-Colorability problem ($3COL$)

INPUT: a finite graph $G = (V, E)$.

QUESTION: Does G have a 3-coloring?

Recall that we only know $3COL \in EXPTIME$ (and $PSPACE$).

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HOWEVER, if we are GIVEN a 3-coloring of G, it is easy (tractable) to VERIFY the correctness of the 3-coloring (and thus know that G is 3-colorable).

Informally, $3COL$ is a projection of a problem in P.

$3COL$ as a projection of a problem in P

Identify $3COL$ with set

$$\{G : 3COL \text{ answers "YES" on input } G\}.$$

Similarly with other decision problems.

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$$3COL\text{-}TEST = \{(G, \chi) : \chi \text{ is a 3-coloring of } G\}.$$

Clearly $3COL\text{-}TEST$ is tractable (in $TIME(N^2)$, hence in P).

And

$$G \in 3COL \quad \Leftrightarrow \quad \exists\chi[(G, \chi) \in 3COL\text{-}TEST].$$

Certificates for $3COL$

If $(G, \chi) \in 3COL\text{-}TEST$, then we call χ a certificate for "$G \in 3COL$."

We say that:

- $3COL\text{-}TEST$ is a polynomial-time certifier for $3COL$.
- $3COL$ is polynomial-time certifiable.
- $3COL$ is in Nondeterministic Polynomial Time (or NP).

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$$L \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$$

$$\begin{array}{c} NP \\ \uplus \\ 3COL \end{array}$$

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More examples of NP problems

The following problems are all in NP (and not known to be in P).

1. $4COL$, $5COL$, etc.

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1. 4*COL*, 5*COL*, etc.
2. *SAT*:
   - INPUT: a boolean formula φ.
   - QUESTION: is φ satisfiable?
   - Certificate: an assignment of values to the variables making φ true.
   - Polynomial-time certifier: given $(\varphi, \mathbf{c})$, decide if $\varphi(\mathbf{c}) = 1$ (i.e., *FVAL*).

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3. ISO:
   - INPUT: two finite graphs G_1, G_2.
   - QUESTION: are G_1 and G_2 isomorphic?
   - Certificate: an isomorphism from G_1 to G_2.
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4. $HAMPATH$:
   - INPUT: a finite directed graph G.
   - QUESTION: does G have a *Hamiltonion path*?

Certifying Turing machines

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To define it precisely, we need the notion of a certifying Turing machine:

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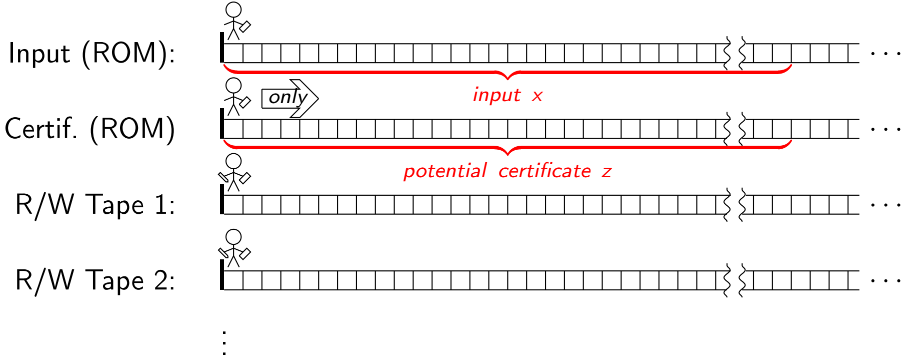

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If $\square$ is a complexity class, then a decision problem D is in $N\square$ iff there exists a decision problem E in two inputs (x, z), and there exists a certifying Turing machine M, such that

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- Exercise: this defines NP equivalently.
- NL =“Nondeterministic $LOGSPACE$”
- $NSPACE$ = “Nondeterministic $PSPACE$”
- $NEXPTIME$ = “Nondeterministic $EXPTIME$”

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$$\begin{aligned} \mathit{PATH\text{-}TEST} \;=\; \{(G,\pi) : & \; G \text{ is a directed graph with } V = \{0,\ldots,n-1\}, \\ & \text{and } \pi = (v_0, v_1, \ldots, v_k) \text{ is a path from 0 to 1 in } G\} \end{aligned}$$

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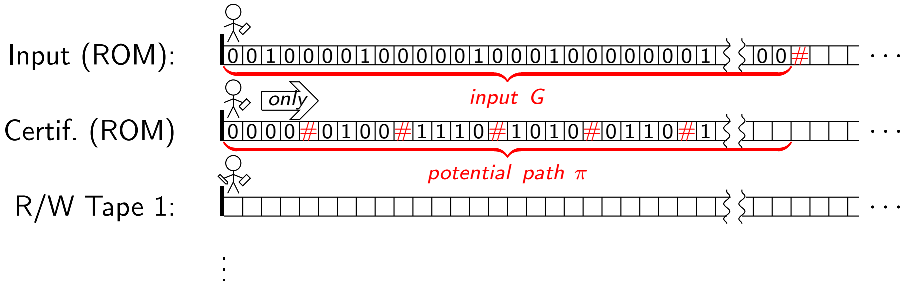
Clearly *PATH* is a projection of *PATH-TEST*.

Certifying *PATH-TEST*

We can build a certifying Turing machine which solves *PATH-TEST* . . .



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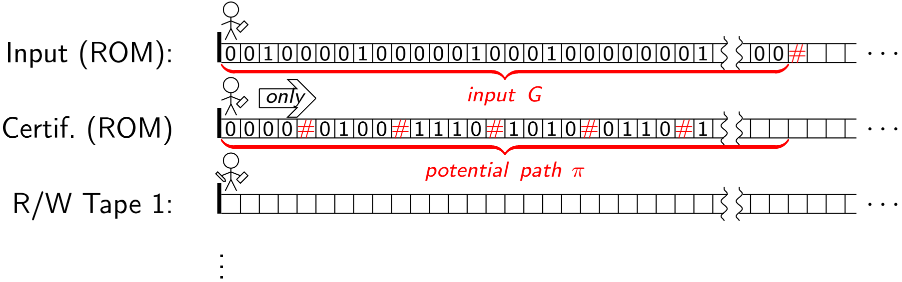

While the certifying student traverses π, the R/W Tape 1 student copies and remembers the last two vertices traversed, and checks the input tape to see if they form an edge.

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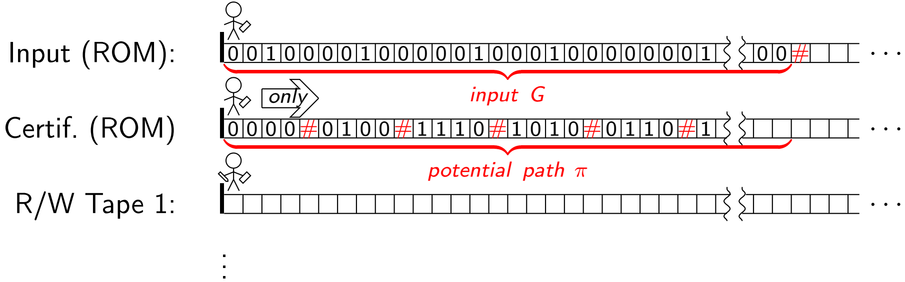

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Only LOGSPACE (as a function of the length of the input G) is needed.

Comparing deterministic and nondeterministic classes

Let $f : \mathbb{N} \to \mathbb{N}$ be "nice" and such that $f(N) \geq \log N$.

Theorem

1. *$TIME(f(N)) \subseteq NTIME(f(N))$ and similarly for SPACE.*

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4. *(Savitch's Theorem): $NSPACE(f(N)) \subseteq SPACE(f(N)^2)$.*

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4. *(Savitch's Theorem): $NSPACE(f(N)) \subseteq SPACE(f(N)^2)$.*

Since $PATH \in NL$, Savitch's theorem shows $PATH \in SPACE((\log N)^2)$.

(Our algorithm showed only that $PATH \in SPACE(\sqrt{N})$.)

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXPTIME \subseteq NEXPTIME$$

L	NL	P	NP	$PSPACE$	$EXPTIME$	$NEXPTIME$
$\rotatebox{90}{$\in$}$	$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$		$\rotatebox{90}{$\in$}$	
$FVAL$, $2COL$	$PATH$		$3COL$, $4COL$, etc. SAT, ISO, $HAMPATH$		CLO	

Summary of complexity classes

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 & & & & & & & & \mathit{NPSPACE} & & & & \\
 & & & & & & & & \| & & & & \\
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 & & \mathit{PATH} & & & & 3\mathit{COL}, & & & & \mathit{CLO} & & \\
\mathit{FVAL}, & & & & & & 4\mathit{COL}, \text{ etc.} & & & & & & \\
2\mathit{COL} & & & & & & \mathit{SAT}, & & & & & & \\
 & & & & & & \mathit{ISO}, & & & & & & \\
 & & & & & & \mathit{HAMPATH} & & & & & &
\end{array}$$

Summary of complexity classes

$$NPSPACE = PSPACE$$

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXPTIME \subseteq NEXPTIME$$

$$NL \neq PSPACE$$

- L ∋ $FVAL$, $2COL$
- NL ∋ $PATH$
- NP ∋ $3COL$, $4COL$, etc., SAT, ISO, $HAMPATH$
- $EXPTIME$ ∋ CLO

Summary of complexity classes

$$\mathit{NPSPACE} = \mathit{PSPACE}$$

$$L \subseteq NL \subseteq P \subseteq NP \subseteq \mathit{PSPACE} \subseteq \mathit{EXPTIME} \subseteq \mathit{NEXPTIME}$$

$$NL \neq \mathit{PSPACE} \qquad P \neq \mathit{EXPTIME}$$

- $L \ni$ FVAL, $2\mathit{COL}$
- $NL \ni$ PATH
- $NP \ni$ $3\mathit{COL}$, $4\mathit{COL}$, etc. SAT, ISO, $\mathit{HAMPATH}$
- $\mathit{EXPTIME} \ni$ CLO

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$NL \neq PSPACE$, $\quad P \neq EXPTIME$, $\quad NP \neq NEXPTIME$

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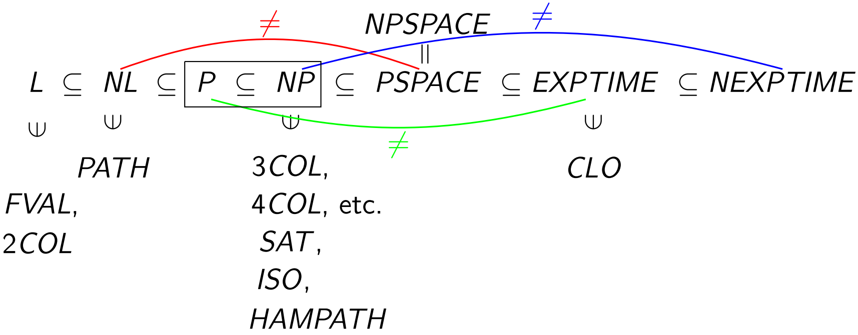

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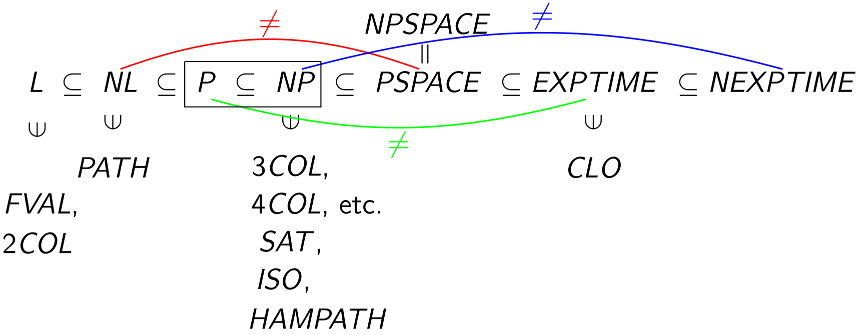


10^6 USD prize (Clay Mathematics Institute) for answering $P \stackrel{?}{=} NP$.

Reductions

Suppose C, D are decision problems.

Suppose $f : C_{inp} \to D_{inp}$ is a function.

We say that

$$f \text{ reduces } C \text{ to } D,$$

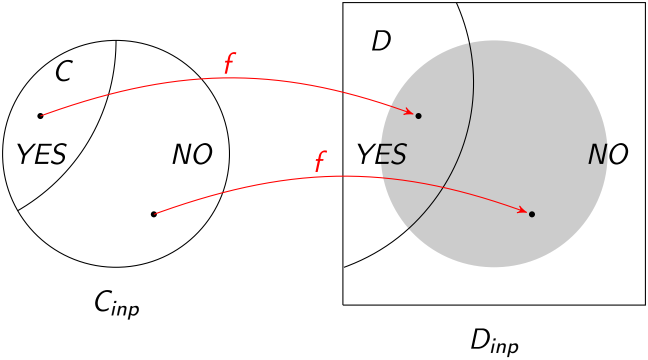
and write

$$C \leq_f D,$$

if for all $x \in C_{inp}$,

$$x \in C \Leftrightarrow f(x) \in D.$$

Picture of $C \leq_f D$



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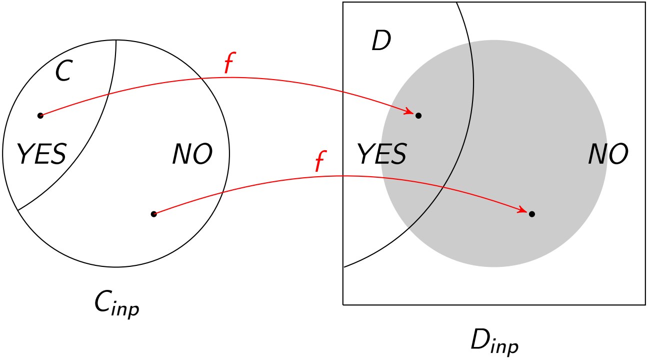

Intuition: if $C \leq_f D$, then

- Algorithms for D and f can be used to solve C.
- Hence D is at least as hard as C (modulo the cost of computing f).

Example

Recall the problems $3COL$ and SAT:

$3COL$

INPUT: a finite graph $G = (V, E)$.
QUESTION: is G 3-colorable?

SAT

INPUT: a boolean formula φ.
QUESTION: is φ satisfiable?

Let's find a function f which reduces $3COL$ to SAT.

A reduction of $3COL$ to SAT

Given a finite graph $G = (V, E)$, we want a boolean formula φ_G such that

$$G \text{ is 3-colorable} \iff \varphi_G \text{ is satisfiable.}$$

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- The variables of φ_G will be all $x_v^{\mathbf{c}}$ $(v \in V,\ \mathbf{c} \in \{\mathbf{r}, \mathbf{g}, \mathbf{b}\})$.
  - Think of $x_v^{\mathbf{c}}$ as representing the assertion "v is colored **c**."

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- For each $v \in V$ let α_v be the formula "v has exactly one color," i.e.,

$$(x_v^{\mathbf{r}} \vee x_v^{\mathbf{g}} \vee x_v^{\mathbf{b}}) \wedge \neg(x_v^{\mathbf{r}} \wedge x_v^{\mathbf{g}}) \wedge \neg(x_v^{\mathbf{r}} \wedge x_v^{\mathbf{b}}) \wedge \neg(x_v^{\mathbf{b}} \wedge x_v^{\mathbf{b}}).$$

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- For $v, w \in V$ let $\beta_{v,w}$ be the formula "v and w have different colors," i.e.,

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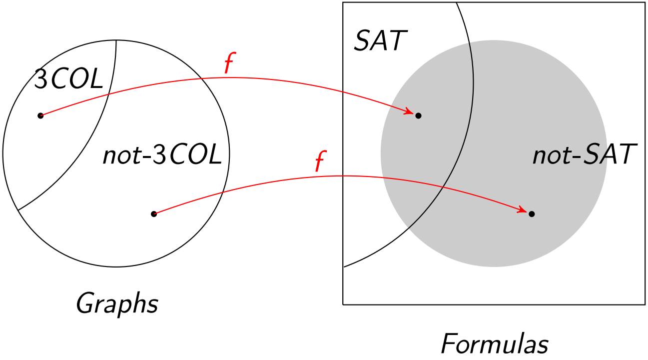
- Let

$$\varphi_G = \left(\bigwedge_{v \in V} \alpha_v \right) \wedge \left(\bigwedge_{(v,w) \in E} \beta_{v,w} \right).$$

This clearly works.

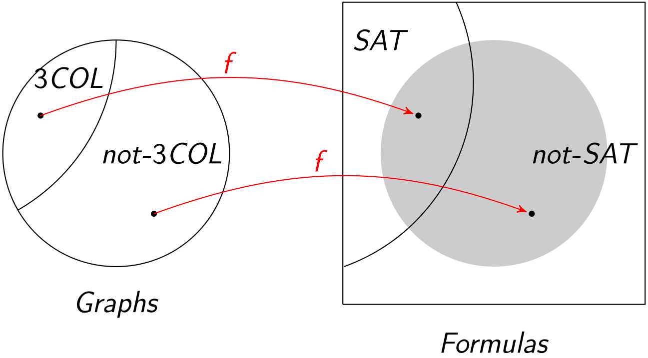
Picture of $3COL \leq_f SAT$

Define $f : G \mapsto \varphi_G$. Then $3COL \leq_f SAT$.



Picture of $3COL \leq_f SAT$

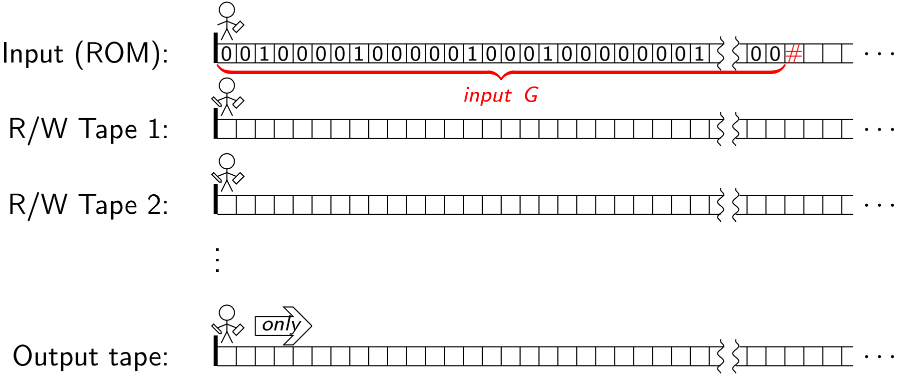
Define $f : G \mapsto \varphi_G$. Then $3COL \leq_f SAT$.



Thus SAT is at least as hard as $3COL$, modulo the cost of computing φ_G.

What is the cost of computing φ_G?

Idea: replace the output bit with an output write-only tape.



At the start.

Computing f with a functional Turing machine

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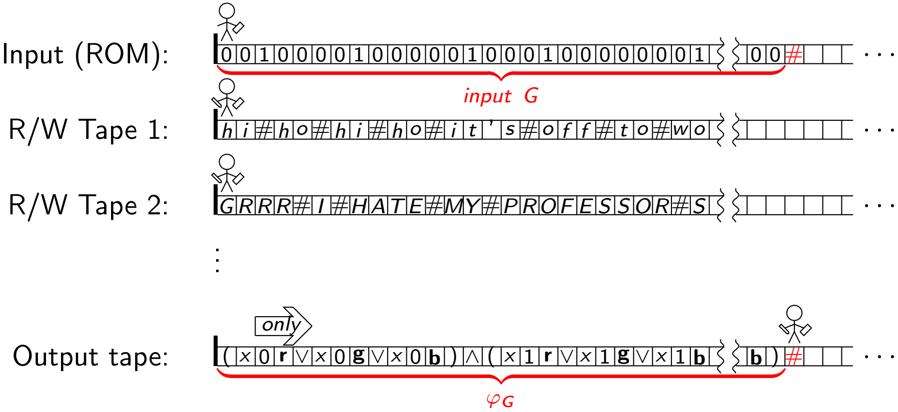

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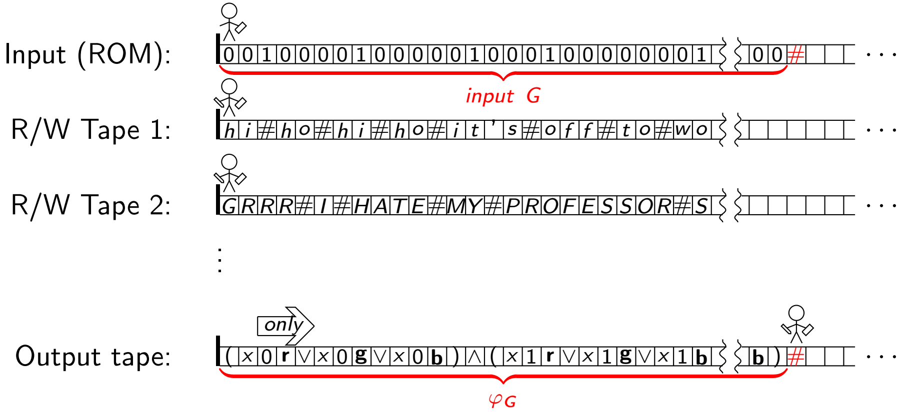

Exercise: Can compute φ_G from G in $TIME(N^2)$ and $SPACE(\log N)$.

Complexity of computing f

In general:

Definition

a functional Turing machine is a Turing machine whose output *bit* is replaced by an output *tape* (write-only).

- Output tape grad student can only move RIGHT.

Let C, D be decision problems with appropriately encoded input sets C_{inp}, D_{inp} respectively.

Definition

A function $f : C_{inp} \to D_{inp}$ is *computed by a functional Turing Machine M* if whenever M is started with input $x \in C_{inp}$, it eventually halts with $f(x)$ written on its output tape.

X-computable functions

Let X be a complexity class (such as P, L, etc.).

Definition

We say that a function $f : C_{inp} \to D_{inp}$ is *computable in* X if there exists a functional Turing Machine which computes f and on input x requires no more resources than those permitted by the definition of X.

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Example: the function $f : G \mapsto \varphi_G$ in our example showing $3COL \leq_f SAT$ is P-computable.

- (In fact, it is L-computable.)

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Lemma

For any decent complexity class X, if $C \leq_f D \in X$ and f is X-computable, then $C \in X$.

X-reductions

Suppose X, Y are complexity classes with $X \subseteq Y$.
Let C, D be decision problems with $C, D \in Y$.

Definition

1. We say that C reduces to D (mod X) and write

$$C \leq_X D$$

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This turns the $\equiv_X$-classes of Y into a poset.

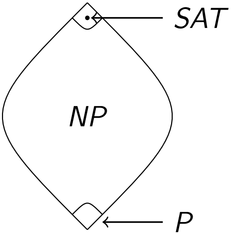
Most widely used when $X = P$.

The picture of NP (mod P)

Theorem

The poset $(NP/\equiv_P, \leq_P)$ has ...

1. *a least element (consisting of all the elements of P), and*
2. *(S. Cook, '71; L. Levin, '73) a greatest element, namely, the $\equiv_P$-class containing SAT.*



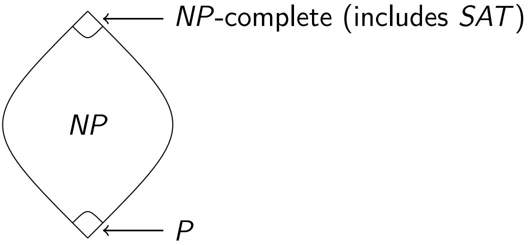
Jargon: SAT is NP-complete (for $\leq_P$ reductions).

Definition

A decision problem D is *NP-complete* if:

- $D \in NP$, and
- $C \leq_P D$ for all $C \in NP$.

Equivalently (by Cook-Levin), D is NP-complete iff $D \equiv_P SAT$.



Karp's Theorem

Theorem (R. Karp, '72)

Many problems are NP-complete.

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Many problems are NP-complete.

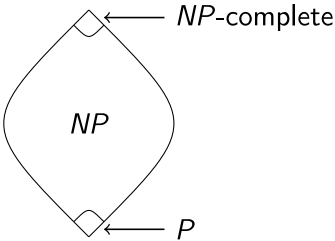
Examples:

- $3COL$, $4COL$, etc.
- $HAMPATH$
- $3SAT$ (the restriction of SAT to formulas in CNF, each conjunct being a disjunction of at most 3 literals)

(Exercise: check that our proof we gave for $3COL \leq_P SAT$ also shows $3COL \leq_P 3SAT$.)

Ladner's Theorem

Remark: the picture below of NP is accurate only if $P \neq NP$:



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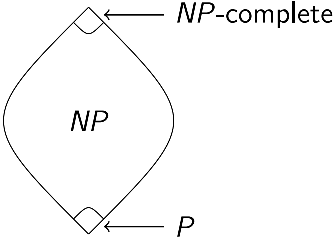

The picture if $P = NP$:  $P = NP = NP$-complete

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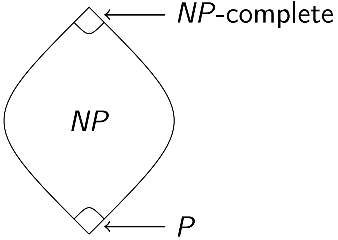

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Theorem (R. Ladner, '75)

If $P \neq NP$, then $|NP/\equiv_P| \geq 3$.

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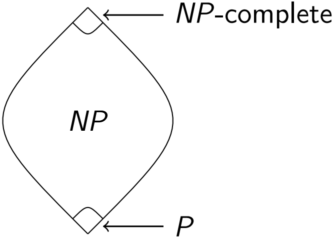

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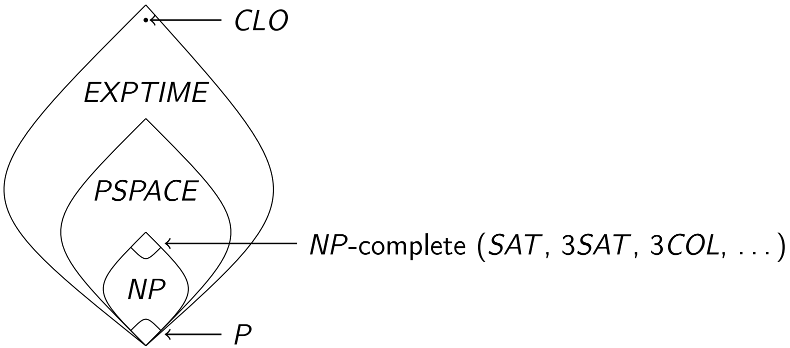
Theorem (R. Ladner, '75)

If $P \neq NP$, then $|NP/\equiv_P| \geq 3$.

In fact, if $P \neq NP$, then $NP/\equiv_P$ is order dense.

The picture of *EXPTIME* (mod *P*)



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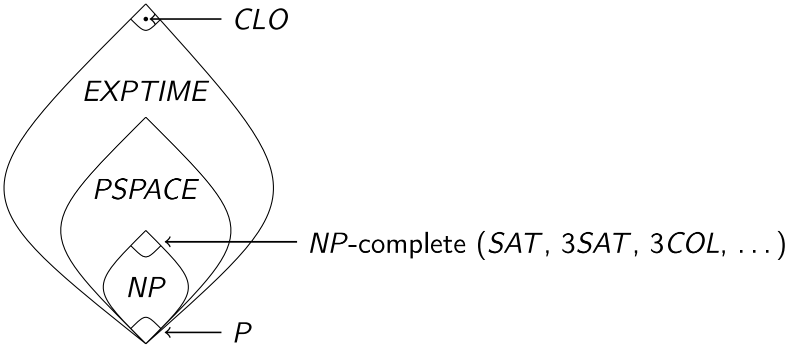

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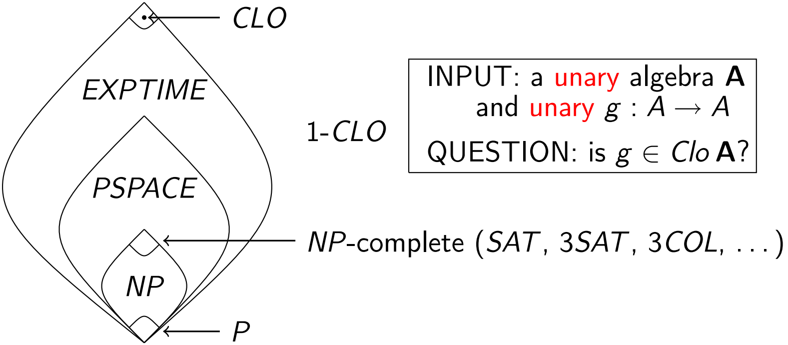

INPUT: a unary algebra $\mathbf{A}$
and unary $g : A \to A$

QUESTION: is $g \in Clo\, \mathbf{A}$?

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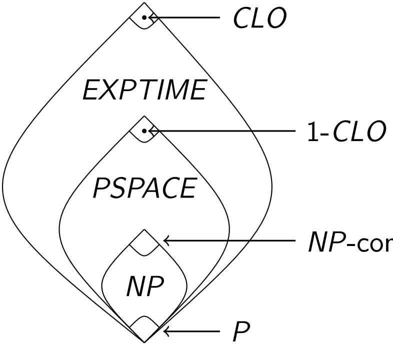

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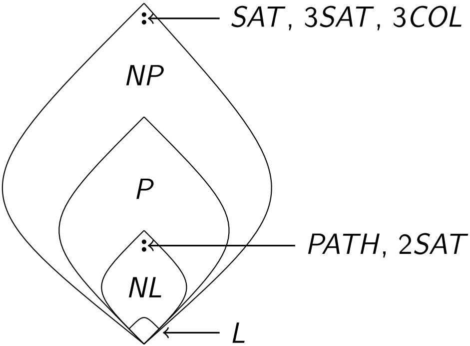
The picture of NP (mod L)



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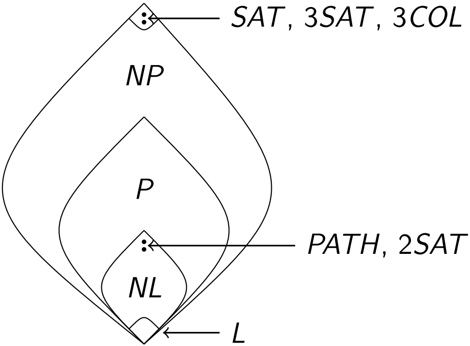

- SAT, $3SAT$ and $3COL$ are NP-complete (for $\leq_L$ reductions).

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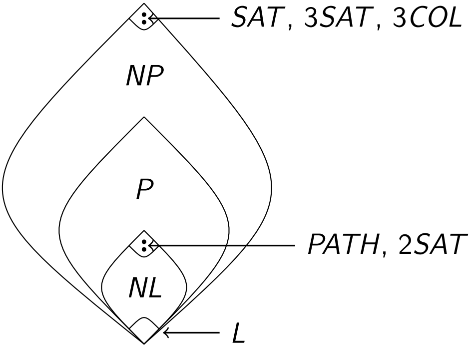

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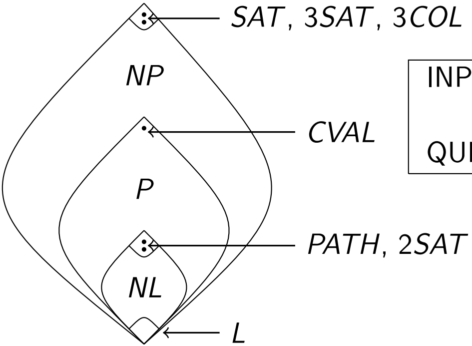

INPUT: a boolean circuit φ
and values $\mathbf{c}$ for variables
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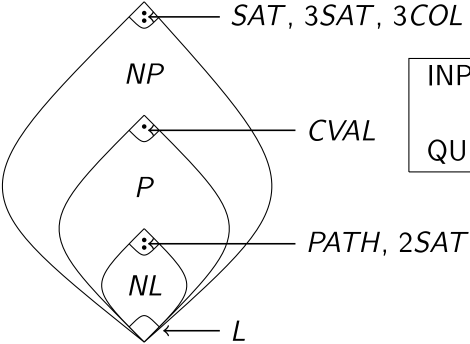

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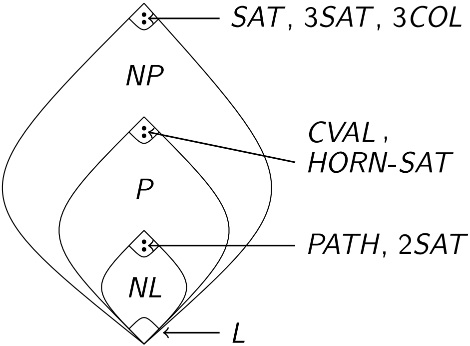
The picture of *NP* (mod *L*)



SAT restricted to CNF φ each of whose clauses has at most one positive literal

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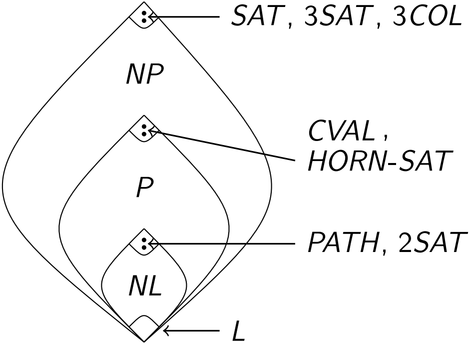

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- (???) $HORN$-SAT and $HORN$-$3SAT$ are also P-complete.

Summary

L	$\subseteq$	NL	$\subseteq$	P	$\subseteq$	NP	$\subseteq$	$PSPACE$	$\subseteq$	$EXPTIME$ $\cdots$
$\Cup$		$\Cup$		$\Cup$		$\Cup$		$\Cup$		$\Cup$
$FVAL$, $2COL$		$PATH$, $2SAT$		$CVAL$, $HORN$-$3SAT$		SAT, $3SAT$, $3COL$, $4COL$, etc. $HAMPATH$		1-CLO		CLO

Moreover, each problem listed above is "hardest in its class," i.e., is complete with respect to either $\leq_P$ or $\leq_L$ reductions.

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Moreover, each problem listed above is "hardest in its class," i.e., is complete with respect to either $\leq_P$ or $\leq_L$ reductions.

In Thursday's lecture: some problems from universal algebra.