Universal Algebra and Computational Complexity
Lecture 1

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Outline

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LECTURE 1: Decision problems and Complexity Classes

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LECTURE 2: Nondeterminism, Reductions and Complete problems

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LECTURE 1: Decision problems and Complexity Classes

LECTURE 2: Nondeterminism, Reductions and Complete problems

LECTURE 3: Results and problems from Universal Algebra

Three themes: problems, algorithms, efficiency

A *Decision Problem* is . . .

- A *YES/NO question*
- parametrized by one or more *inputs*.
  - Inputs must:
    - range over an *infinite* class.
    - be "finitistically described"

Three themes: problems, algorithms, efficiency

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What we seek:

- An *algorithm* which correctly answers the question for all possible inputs.

Three themes: problems, algorithms, efficiency

A *Decision Problem* is . . .

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- parametrized by one or more *inputs*.
  - Inputs must:
    - range over an *infinite* class.
    - be "finitistically described"

What we seek:

- An *algorithm* which correctly answers the question for all possible inputs.

What we ask:

- How *efficient* is this algorithm?
- Is there a better (more efficient) algorithm?

Directed Graph Reachability problem (PATH)

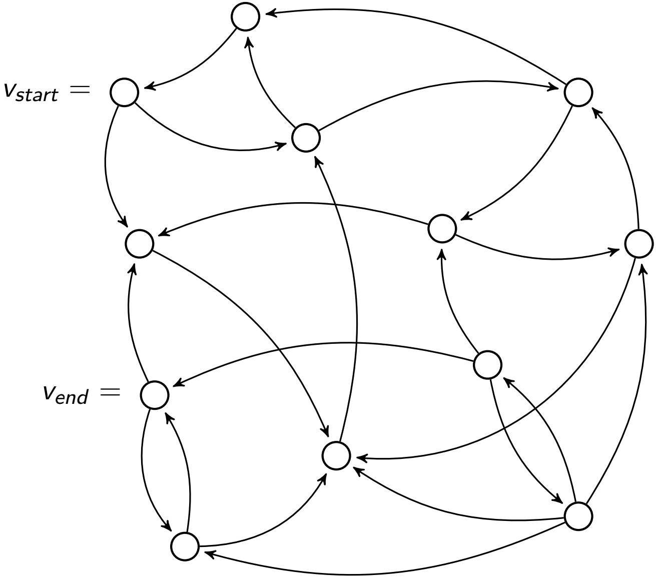
INPUT:

- A finite directed graph $G = (V, E)$
- Two distinguished vertices $v_{start}, v_{end} \in V$.

QUESTION:

- Does there exist in G a *directed path* from v_{start} to v_{end}?

An Algorithm for PATH



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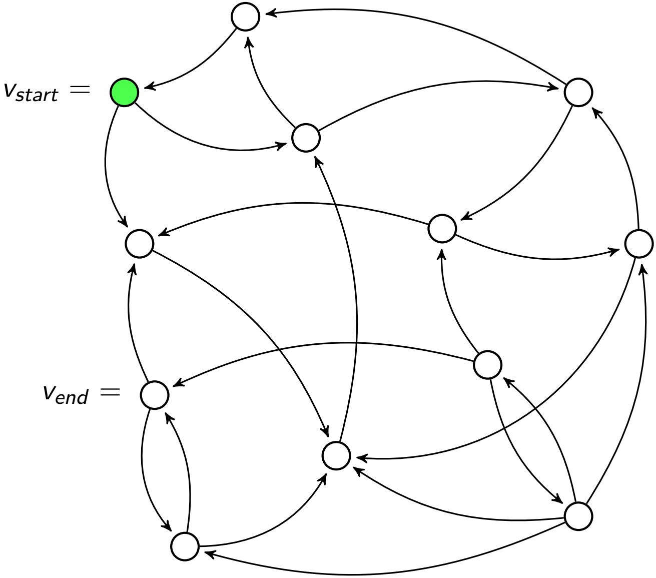

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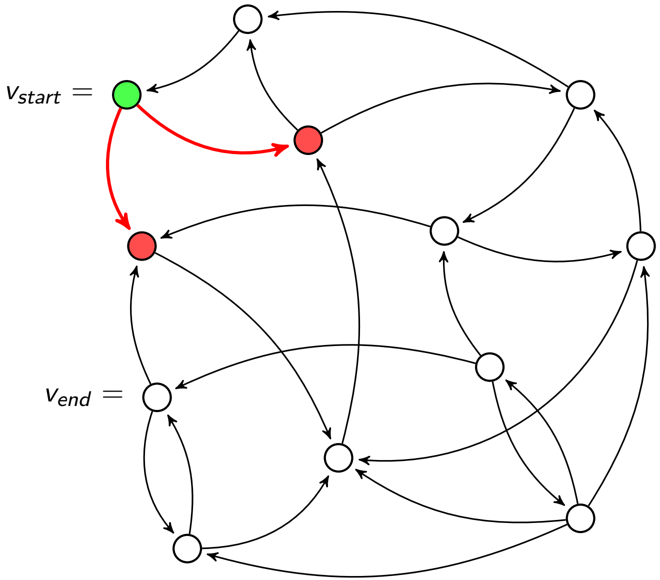

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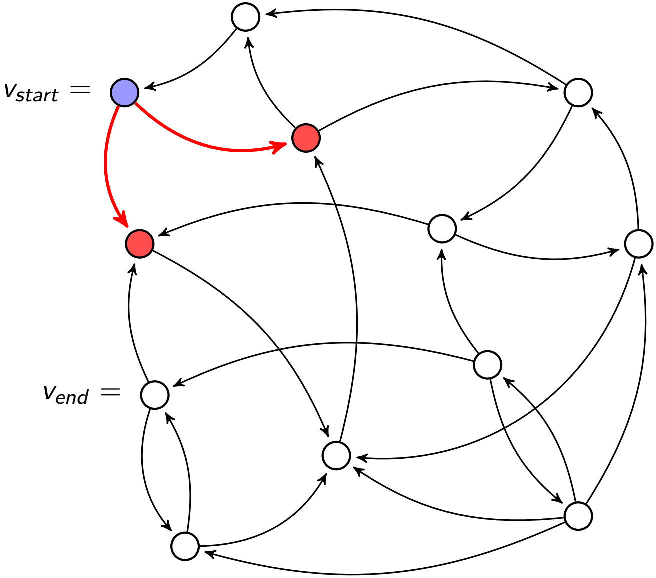

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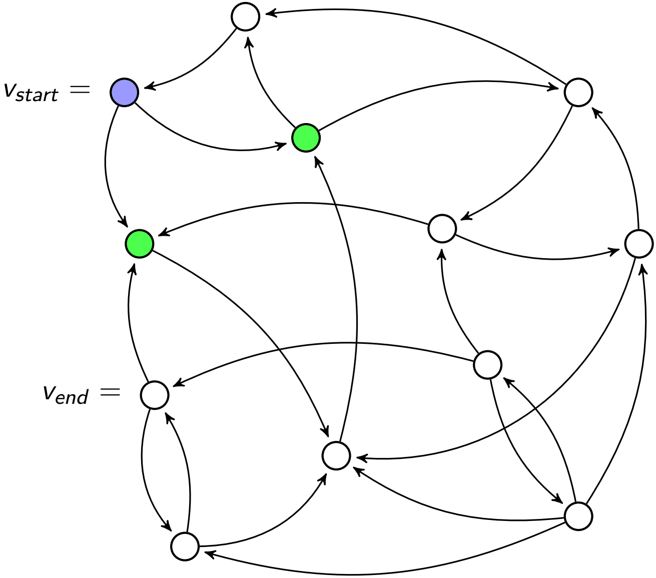

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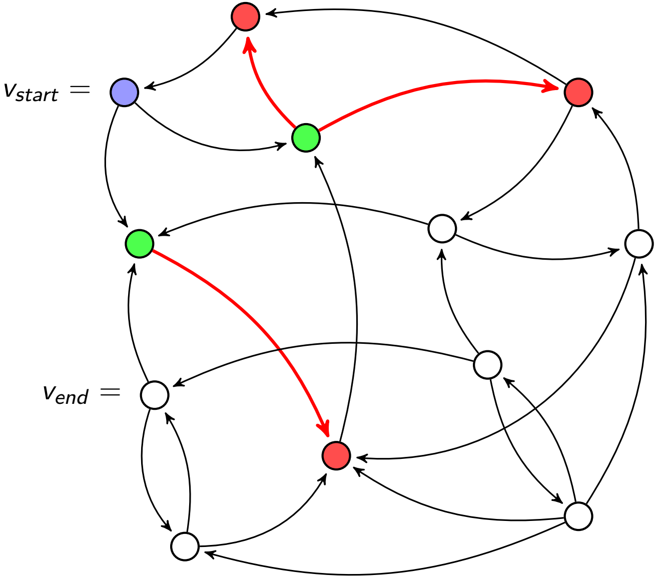

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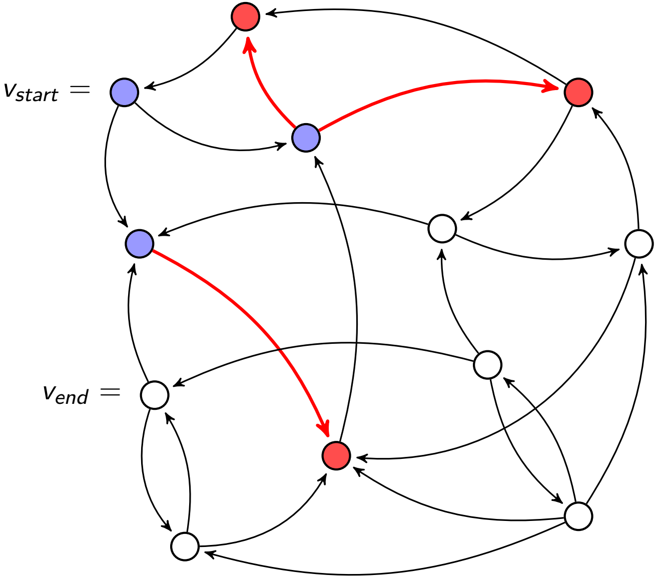
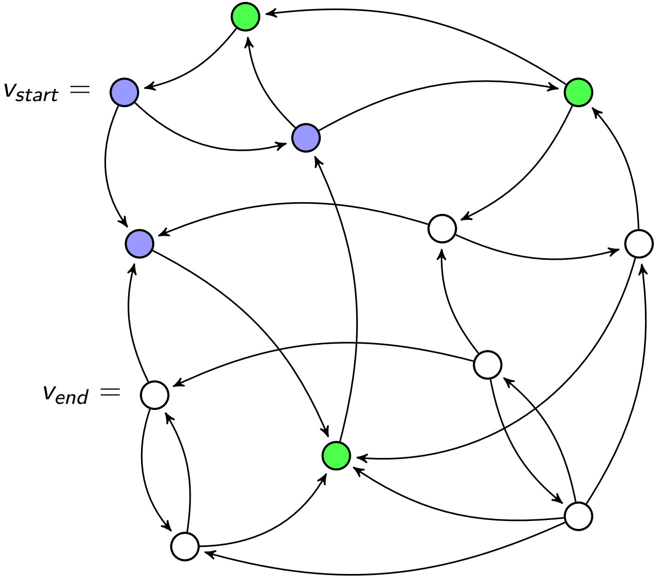


An Algorithm for PATH



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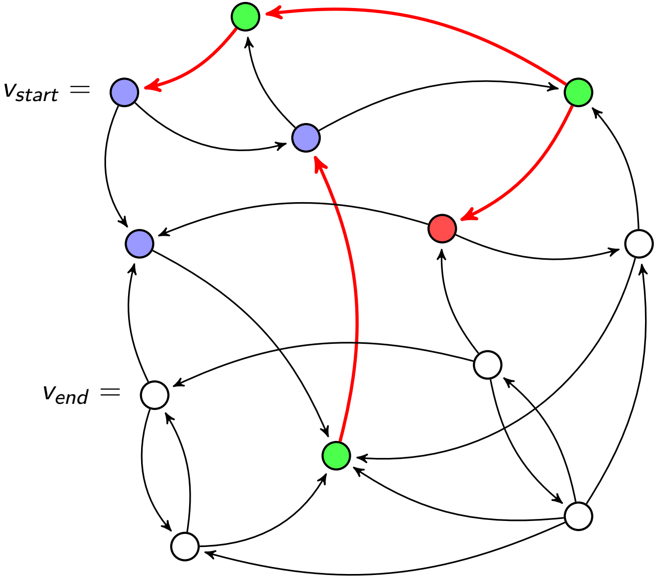

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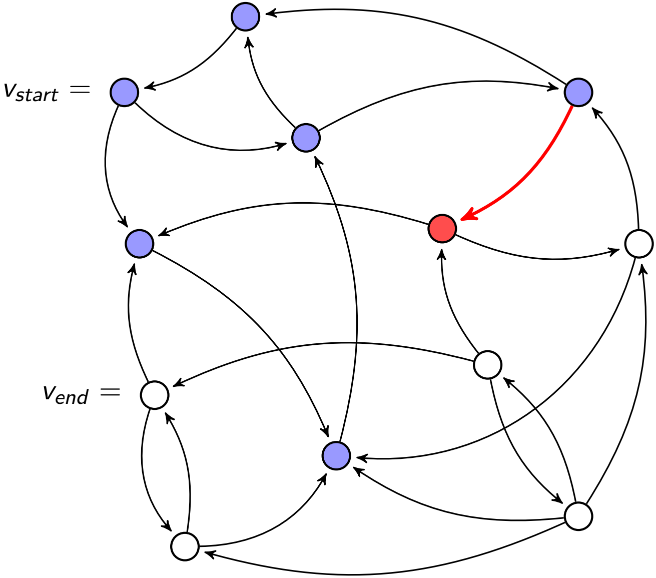

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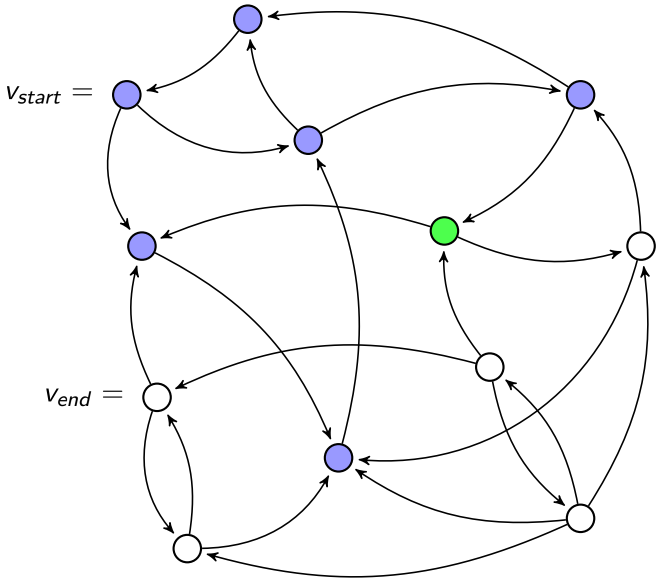

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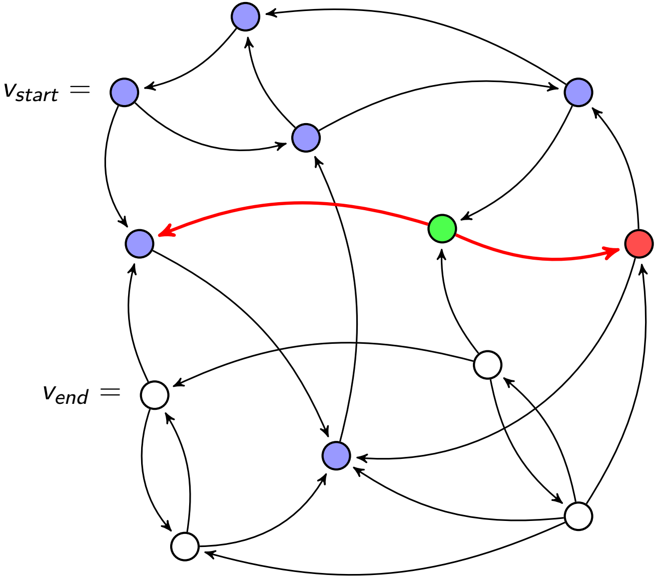

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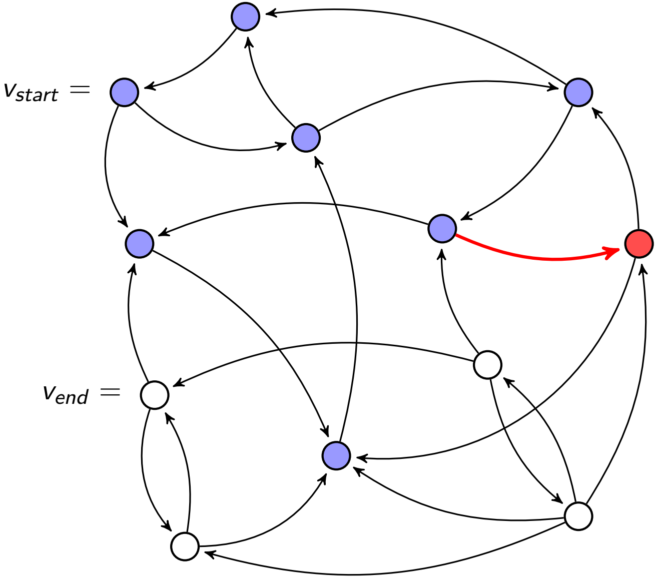
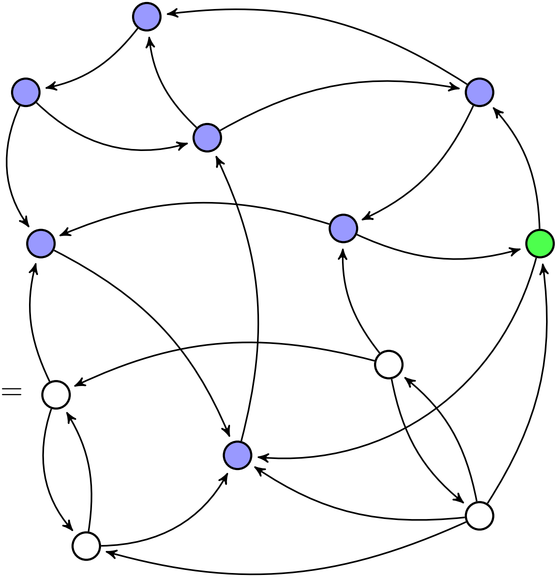


An Algorithm for PATH

$v_{start} =$

$v_{end} =$



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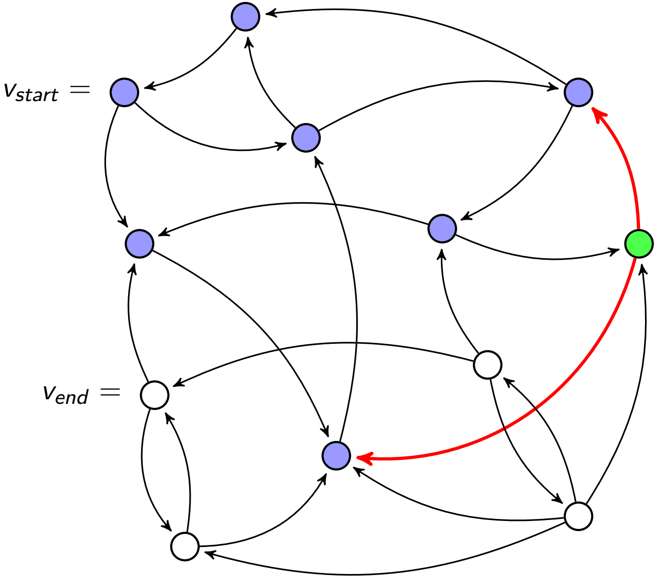

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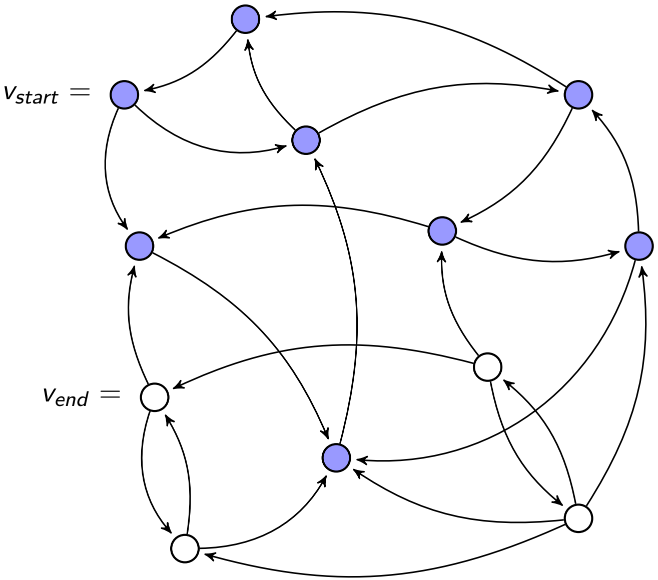
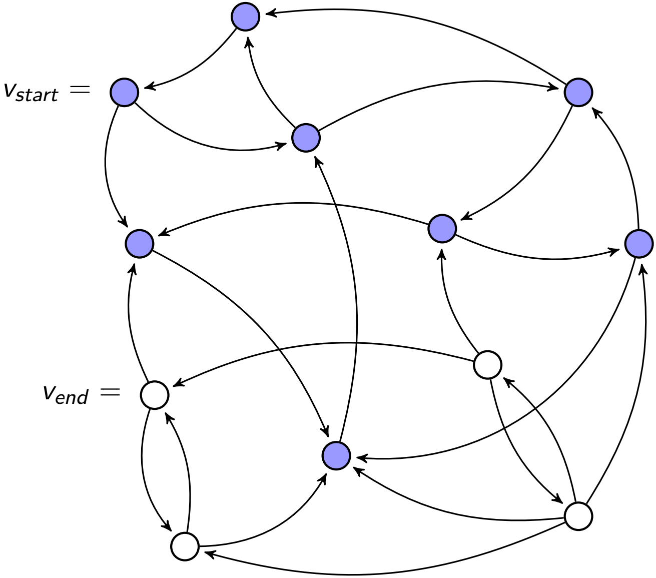


An Algorithm for PATH



Answer: "NO"

Efficiency of this algorithm

How long does this algorithm take?

- I.e., how many steps . . .

Efficiency of this algorithm

How long does this algorithm take?

- I.e., how many steps ...
- ... as a function of the size of the input graph.

Efficiency of this algorithm

How long does this algorithm take?

- I.e., how many steps . . .
- . . . as a function of the size of the input graph.

I'll give three answers to this.

First answer – Heuristics

Only action is changing a vertex's color.

Only changes possible are

- white $\Rightarrow$ red
- red $\Rightarrow$ green
- green $\Rightarrow$ blue.

So if $n = |V|$, then the algorithm requires at most $3n$ vertex-color changes.

Second answer – pseudo-code

Simplifying assumptions:

- $V = \{0, 1, \ldots, n-1\}$
- E is encoded by the adjacency matrix $M_E = [e_{i,j}]$ where

$$e_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{else.} \end{cases}$$

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- E is encoded by the adjacency matrix $M_E = [e_{i,j}]$ where

$$e_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{else.} \end{cases}$$

Auxiliary variables:

- i, j will range over $\{0, 1, \ldots, n{-}1\}$.
- For $i < n$ let c_i be a variable recording the color of vertex i.
- Let *GreenVar* be a variable storing whether there are green-colored vertices.

Second answer – pseudo-code

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n - 1$ set $c_i := \mathit{white}$.
- Set $c_{start} = \mathit{green}$.
- Set $\mathit{GreenVar} := \mathit{yes}$.

Second answer – pseudo-code

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n - 1$ set $c_i :=$ *white*.
- Set $c_{start} =$ *green*.
- Set *GreenVar* := *yes*.
- MAIN LOOP: While *GreenVar* = *yes* do:
    - For $i = 0$ to $n - 1$; for $j = 0$ to $n - 1$
        - if $e_{i,j} = 1$ and $c_i =$ *green* and $c_j =$ *white* then set $c_j :=$ *red*.
    - For $i = 0$ to $n - 1$
        - If $c_i =$ *green* then set $c_i :=$ *blue*
    - Set *GreenVar* := *no*
    - For $i = 0$ to $n - 1$
        - If $c_i =$ *red* then (set $c_i :=$ *green* and set *GreenVar* := *yes*)

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n - 1$ set $c_i :=$ *white*.
- Set $c_{start} =$ *green*.
- Set *GreenVar* := *yes*.
- MAIN LOOP: While *GreenVar* = *yes* do:
  - For $i = 0$ to $n - 1$; for $j = 0$ to $n - 1$
    - if $e_{i,j} = 1$ and $c_i =$ *green* and $c_j =$ *white* then set $c_j :=$ *red*.
  - For $i = 0$ to $n - 1$
    - If $c_i =$ *green* then set $c_i :=$ *blue*
  - Set *GreenVar* := *no*
  - For $i = 0$ to $n - 1$
    - If $c_i =$ *red* then (set $c_i :=$ *green* and set *GreenVar* := *yes*)
- If $c_{end} =$ *blue* then output YES; else output NO.

Second answer – pseudo-code

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n-1$ set $c_i := white$.
- Set $c_{start} = green$.
- Set $GreenVar := yes$.
- MAIN LOOP: While $GreenVar = yes$ do:
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$.
  - For $i = 0$ to $n-1$
    - If $c_i = green$ then set $c_i := blue$
  - Set $GreenVar := no$
  - For $i = 0$ to $n-1$
    - If $c_i = red$ then (set $c_i := green$ and set $GreenVar := yes$)
- If $c_{end} = blue$ then output YES; else output NO.

n loops

Second answer – pseudo-code

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n-1$ set $c_i := white$.
- Set $c_{start} = green$.
- Set $GreenVar := yes$.
- MAIN LOOP: While $GreenVar = yes$ do:
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$.
  - For $i = 0$ to $n-1$
    - If $c_i = green$ then set $c_i := blue$
  - Set $GreenVar := no$
  - For $i = 0$ to $n-1$
    - If $c_i = red$ then (set $c_i := green$ and set $GreenVar := yes$)
- If $c_{end} = blue$ then output YES; else output NO.

n loops

n^2 cases

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n-1$ set $c_i := white$.
- Set $c_{start} = green$.
- Set $GreenVar := yes$.
- MAIN LOOP: While $GreenVar = yes$ do:
    - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$
        - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$.
    - For $i = 0$ to $n-1$
        - If $c_i = green$ then set $c_i := blue$
    - Set $GreenVar := no$
    - For $i = 0$ to $n-1$
        - If $c_i = red$ then (set $c_i := green$ and set $GreenVar := yes$)
- If $c_{end} = blue$ then output YES; else output NO.

n loops

n^2 cases

$\boxed{O(n^3) \text{ steps if } n = |V|}$

Third answer – machine implementation

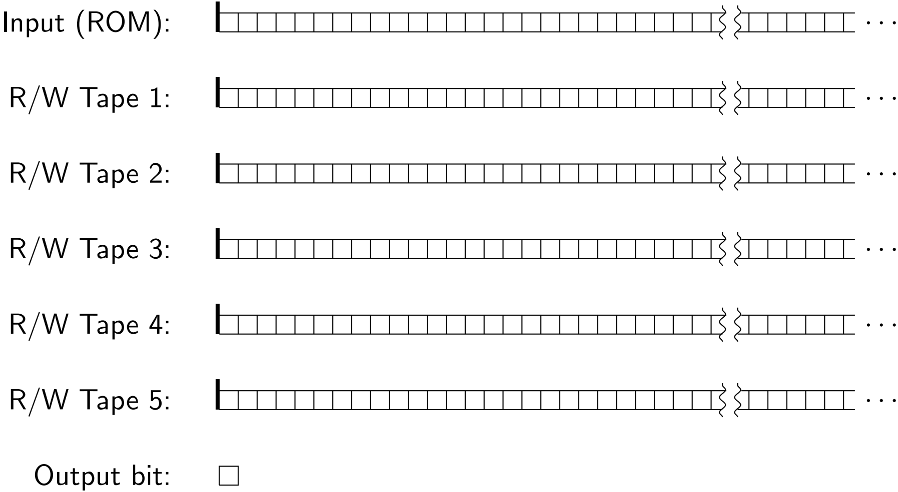
Again assume $V = \{0, 1, \ldots, n-1\}$.

Assume also that $v_{start} = 0$ and $v_{end} = 1$.

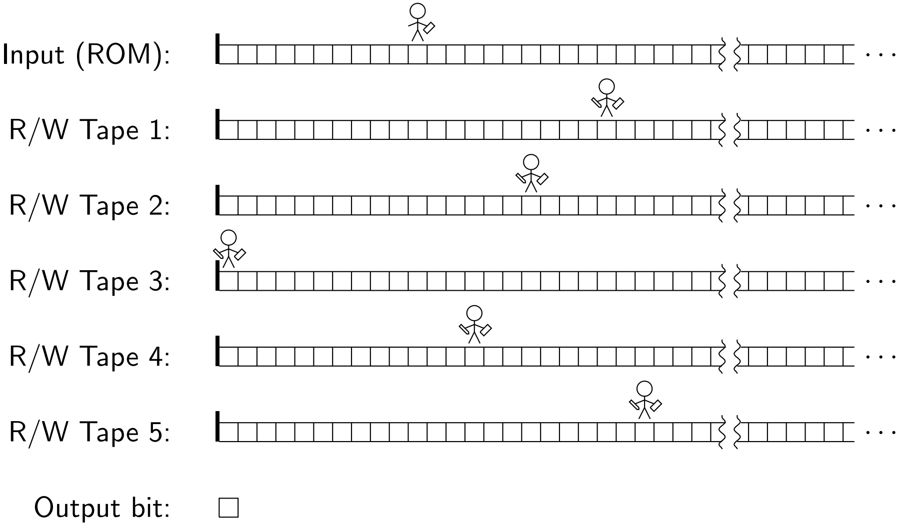
Assume the adjacency matrix is presented as a binary string of length n^2.

Implement the algorithm on a *Turing machine*.

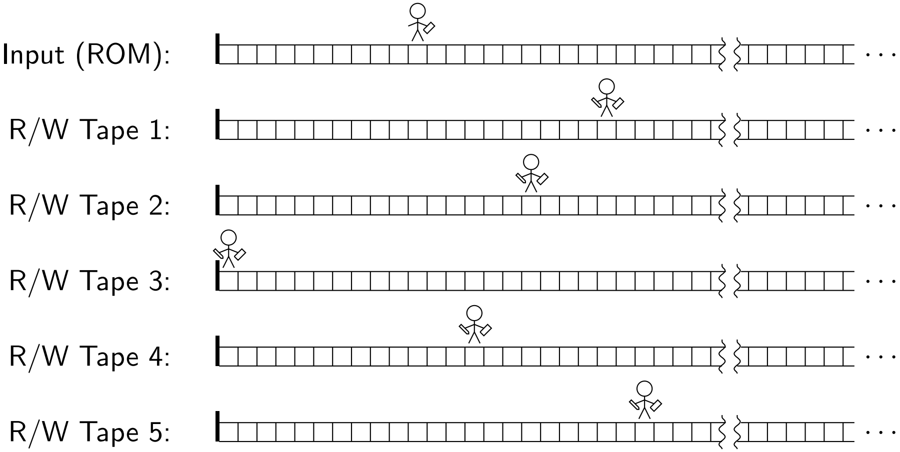
Turing machine



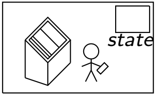
Turing machine



Turing machine



Output bit: ☐



Tape	In	1	2	3	4	5
char						
1^{st}?						

Turing machine

Input (ROM):

R/W Tape 1:

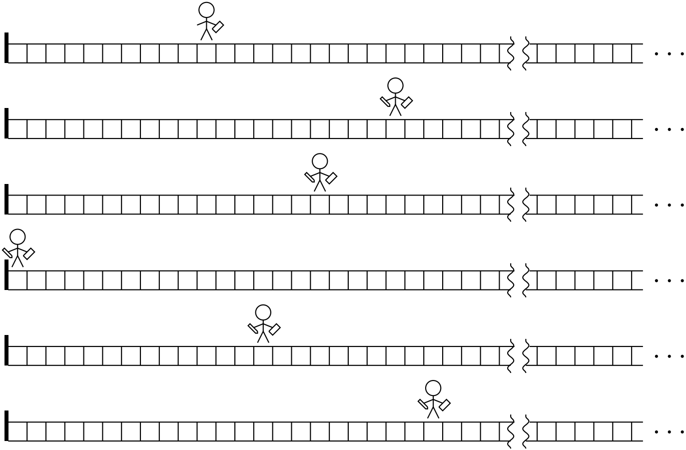
R/W Tape 2:

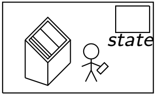
R/W Tape 3:

R/W Tape 4:

R/W Tape 5:

Output bit:





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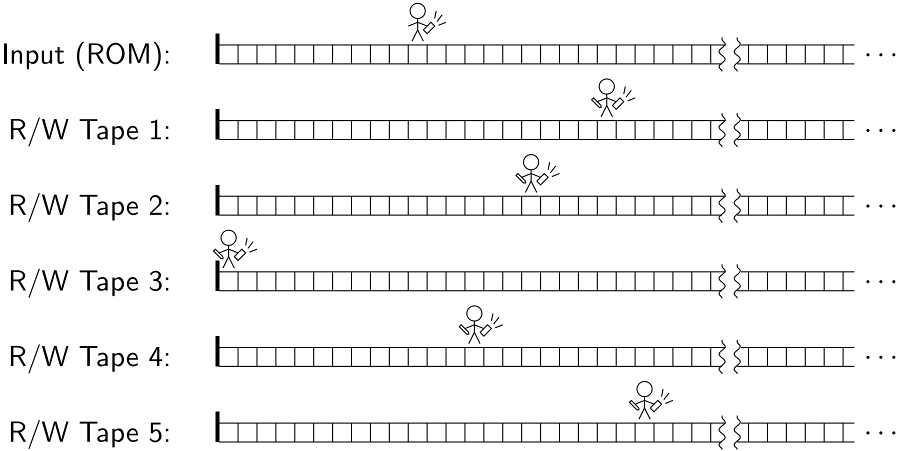

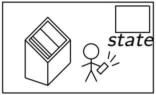

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Turing machine

Input (ROM):

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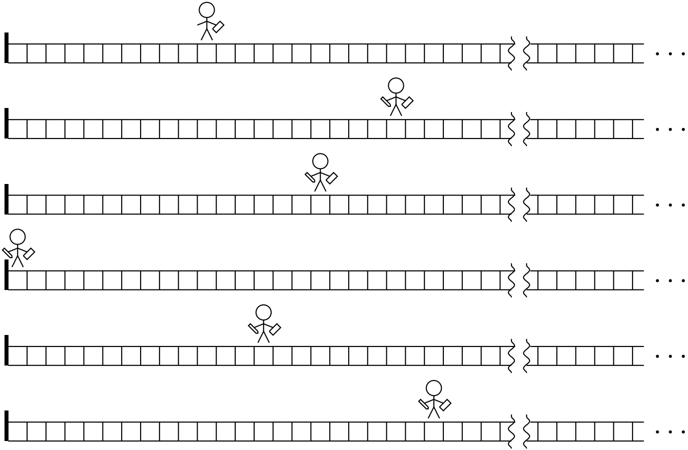
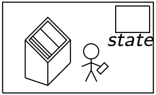
R/W Tape 5:

Output bit:





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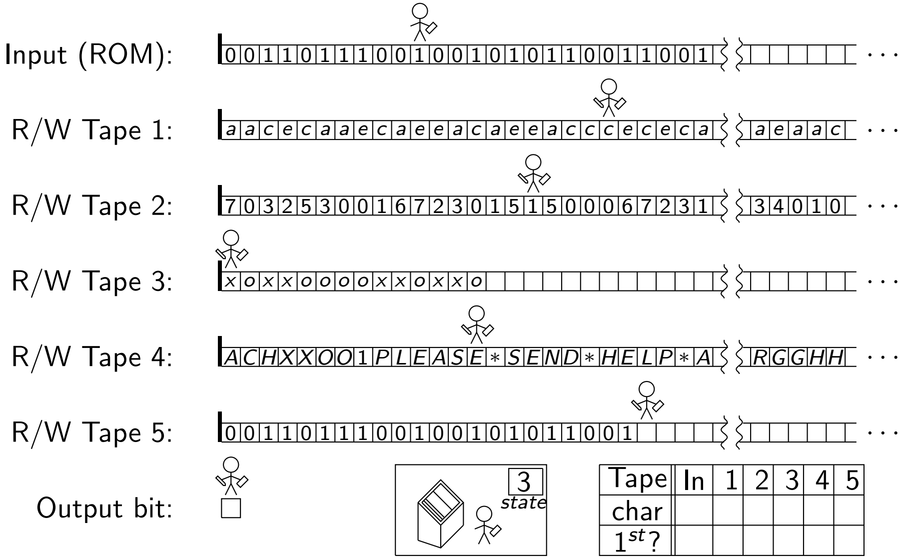
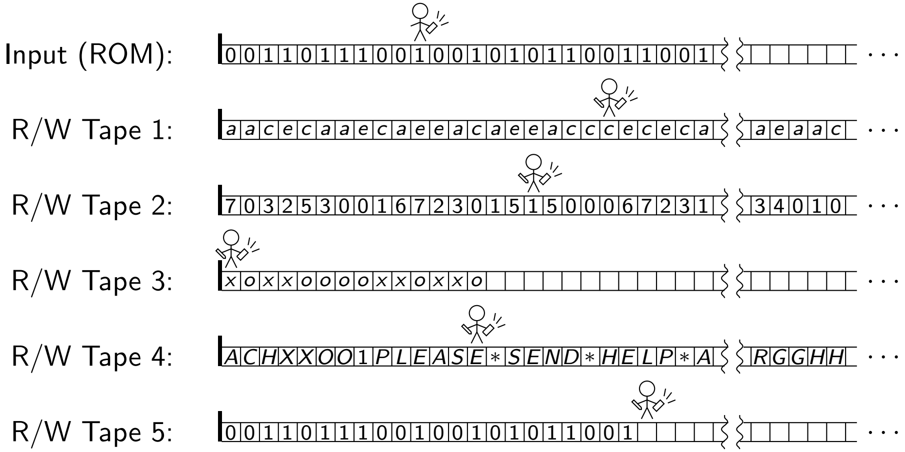
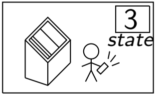
Turing machine



Turing machine

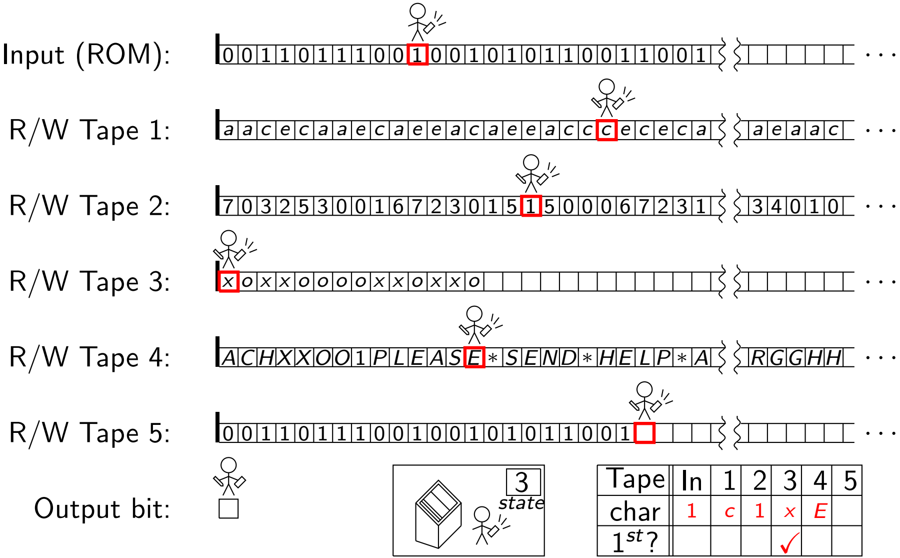


Output bit:



Tape	In	1	2	3	4	5
char						
1^{st}?						

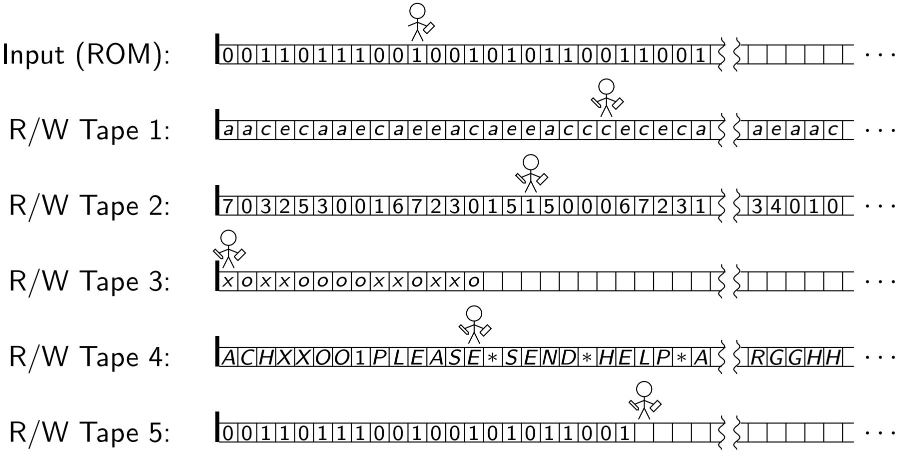
Prof asks for status

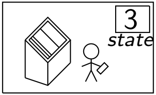


Tape	In	1	2	3	4	5
char	1	c	1	x	E	
1^{st}?				✓		

Students send reports

Turing machine

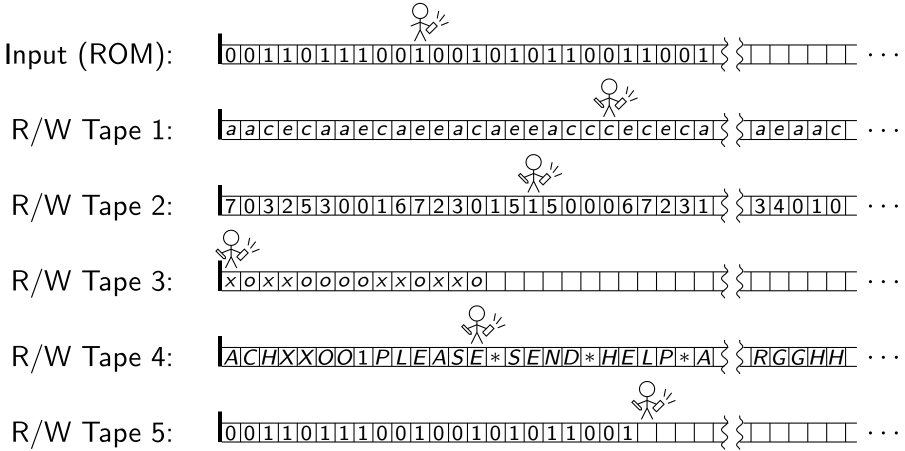




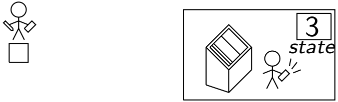
Tape	In	1	2	3	4	5
char	1	*c*	1	*x*	*E*	
1^{st}?				✓		

Prof consults manual

Turing machine



Output bit:



Tape	In	1	2	3	4	5
char	1	c	1	x	E	
1^{st}?				✓		

Prof sends instructions

Turing machine

Input (ROM): 0 0 1 1 0 1 1 1 0 0 1 0 0 1 0 1 0 1 1 0 0 1 1 0 0 1 ⋯

R/W Tape 1: a a c e c a a e c a e e a c a e e a c c c e c e c a ⋯ a e a a c ⋯

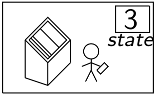
R/W Tape 2: 7 0 3 2 5 3 0 0 1 6 7 2 3 0 1 5 1 5 0 0 0 6 7 2 3 1 ⋯ 3 4 0 1 0 ⋯

R/W Tape 3: x o x x o o o o x x o x x o ⋯

R/W Tape 4: A C H X X O O 1 P L E A S $\boxed{O}$ * S E N D * H E L P * A ⋯ R G G H H ⋯

R/W Tape 5: 0 0 1 1 0 1 1 1 0 0 1 0 0 1 0 1 0 1 1 0 0 1 ⋯

Output bit: □



Tape	In	1	2	3	4	5
char	1	c	1	x	E	
1^{st}?				✓		

Action 1: 4 writes “O”.

Turing machine

Input (ROM):

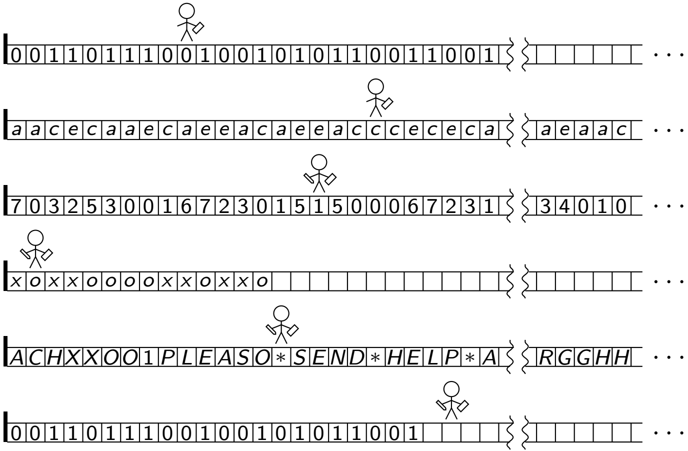
R/W Tape 1:

R/W Tape 2:

R/W Tape 3:

R/W Tape 4:

R/W Tape 5:



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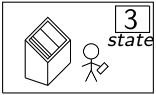
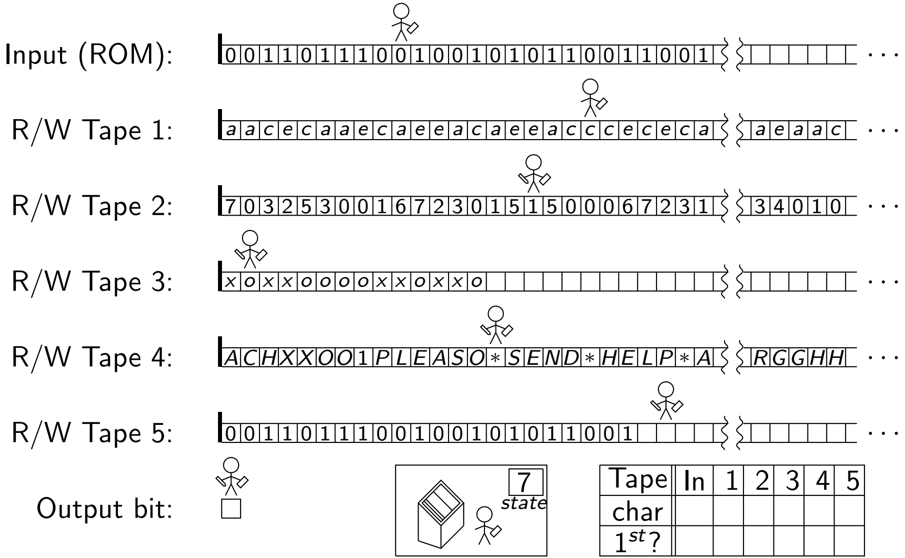


Tape	In	1	2	3	4	5
char	1	c	1	x	E	
1^{st}?				✓		

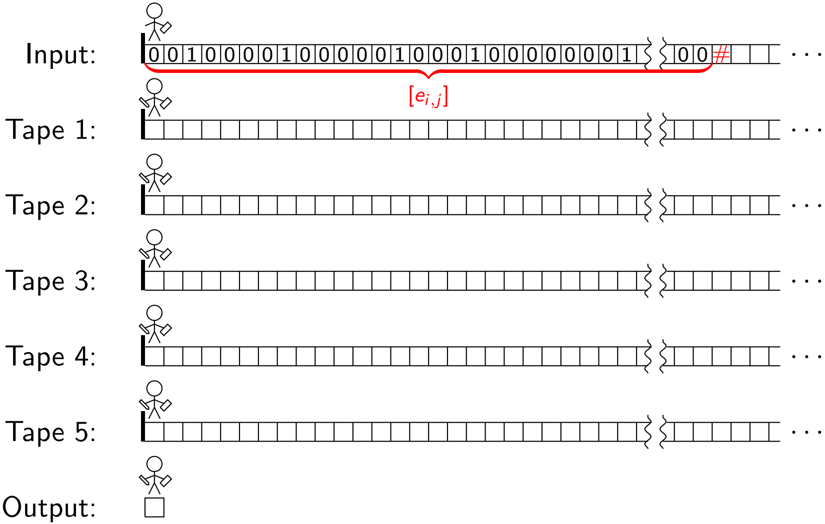
Action 2: Students move as directed

Turing machine

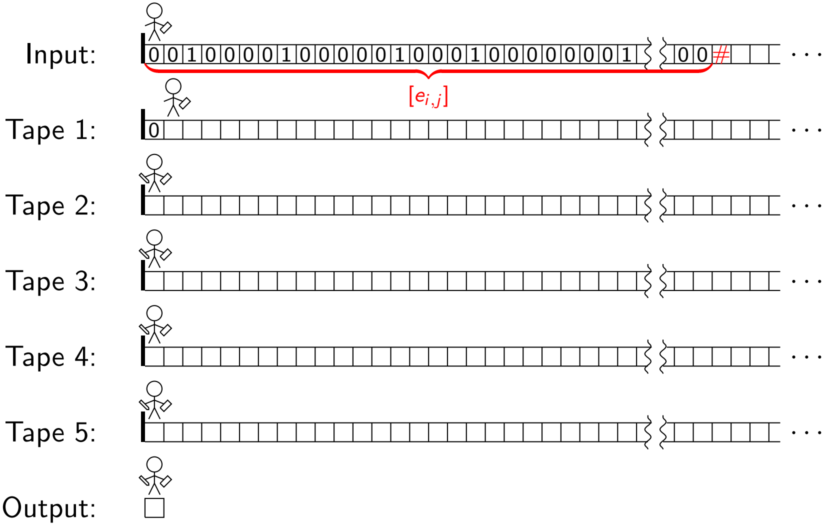


Action 3: Update state, ready for next "step"

Implementing the algorithm for *PATH*



Implementing the algorithm for *PATH*



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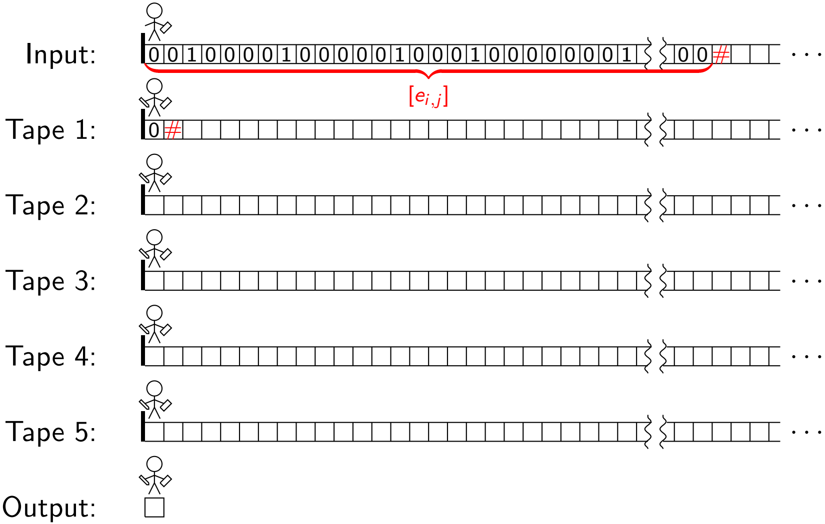
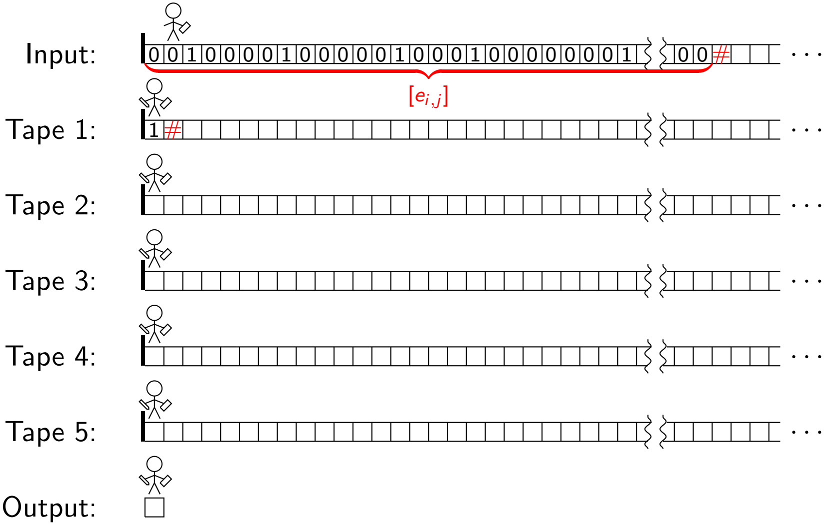
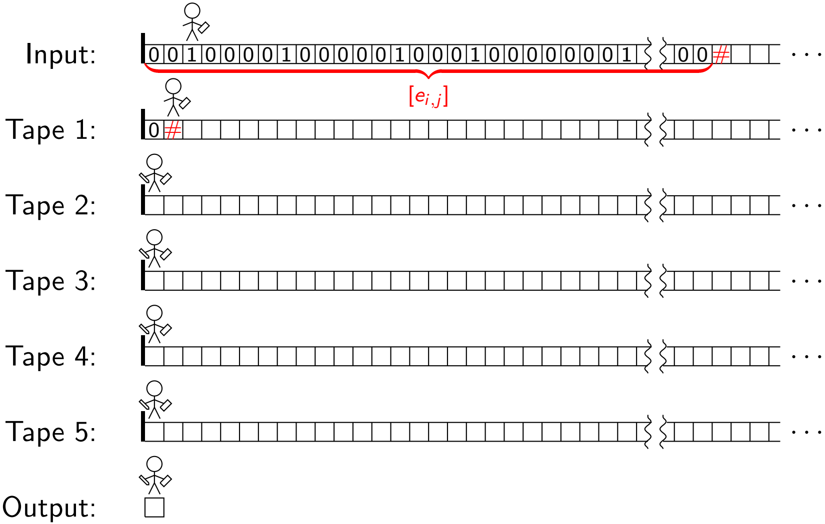


Implementing the algorithm for *PATH*



Implementing the algorithm for *PATH*



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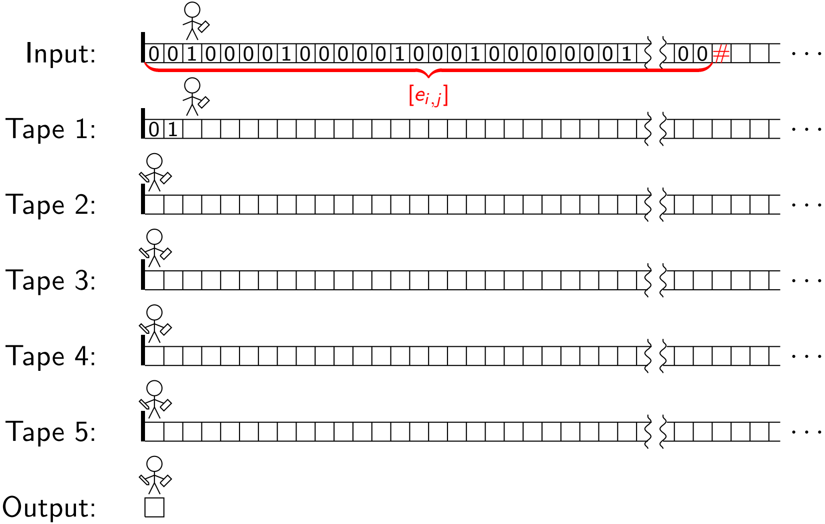

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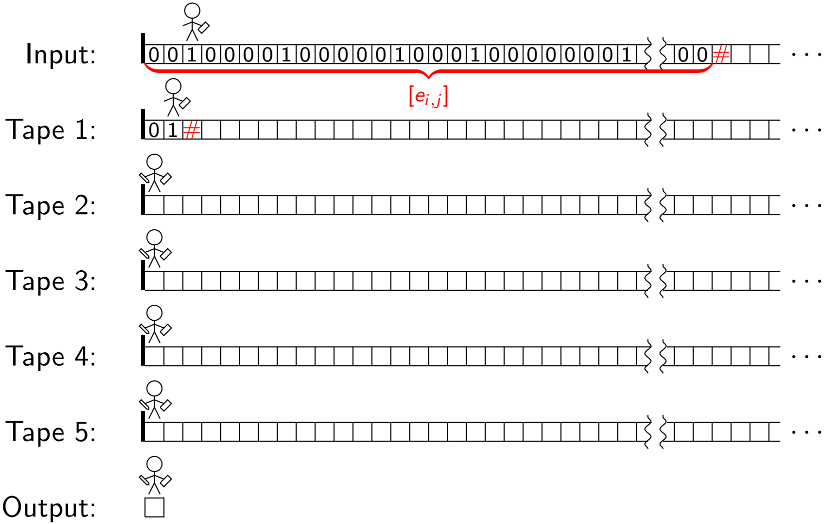

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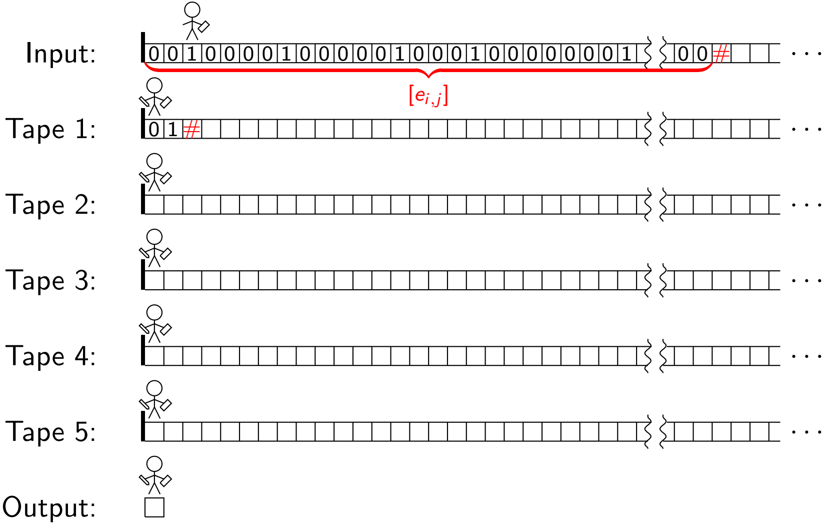

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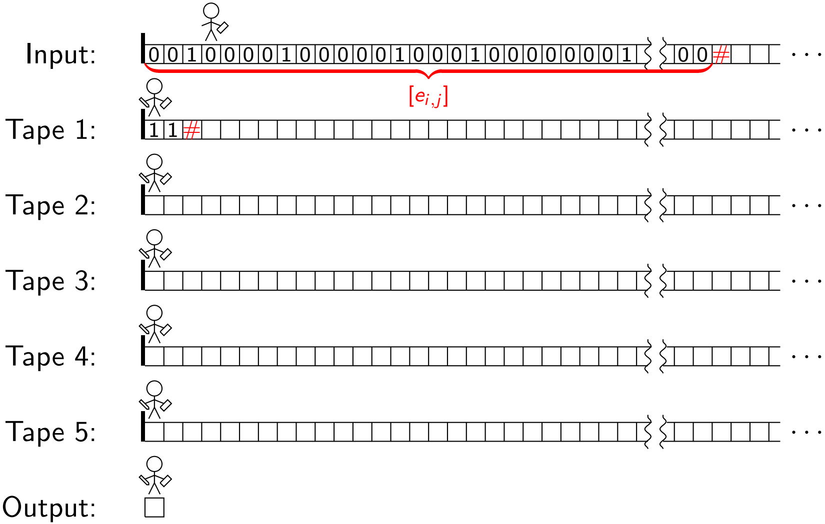

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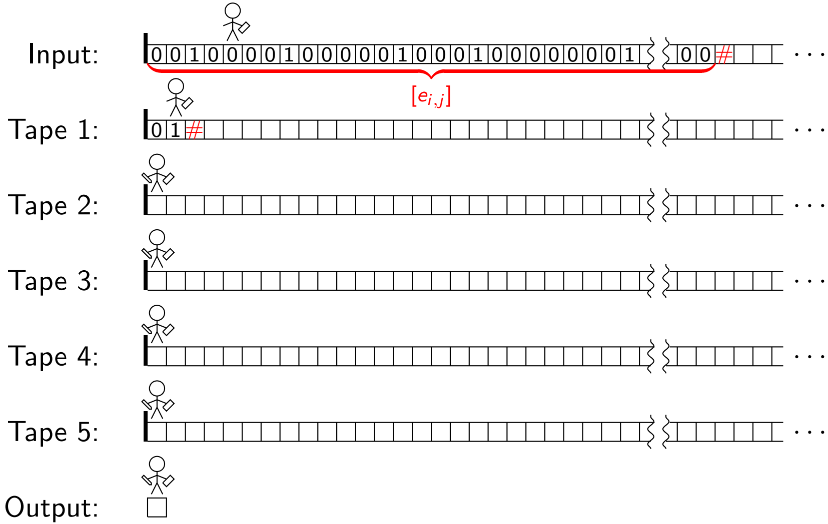

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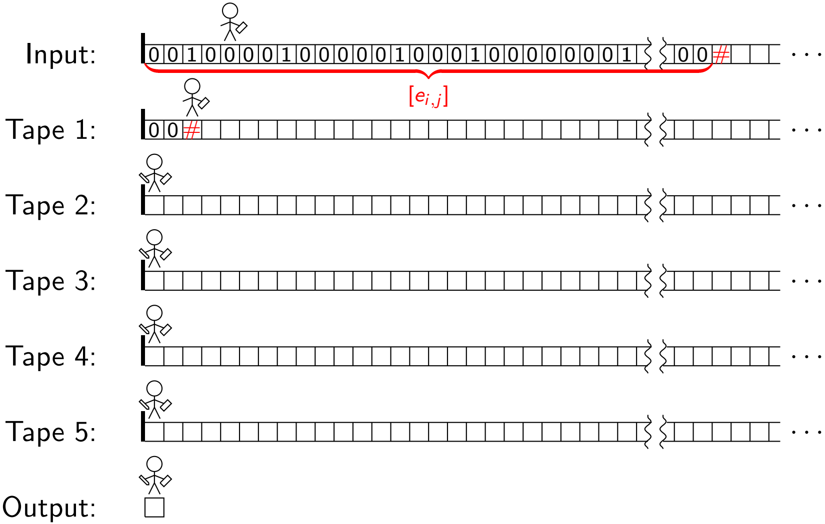

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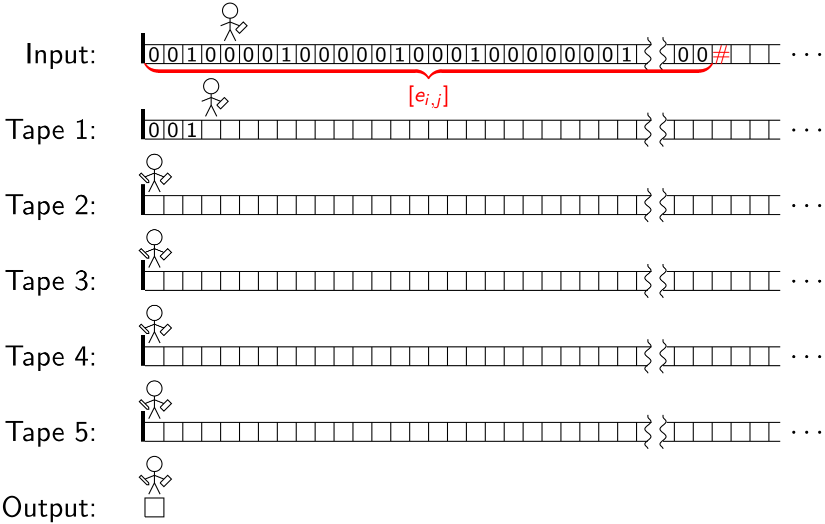

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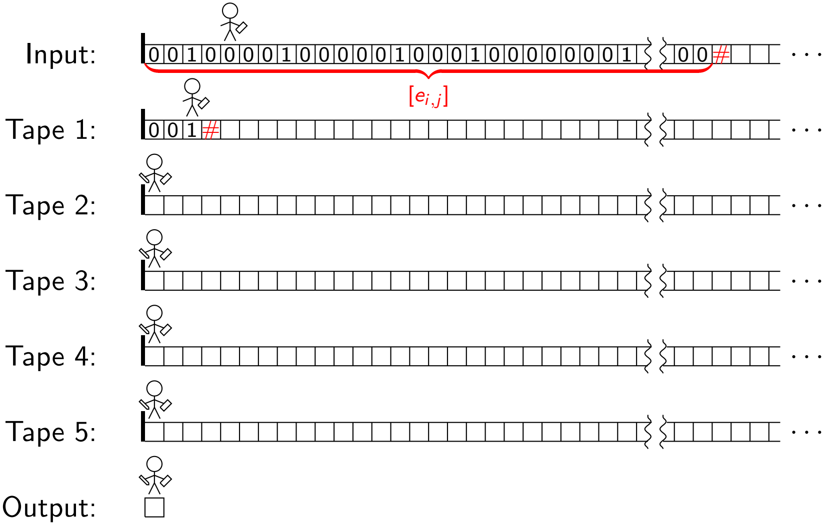

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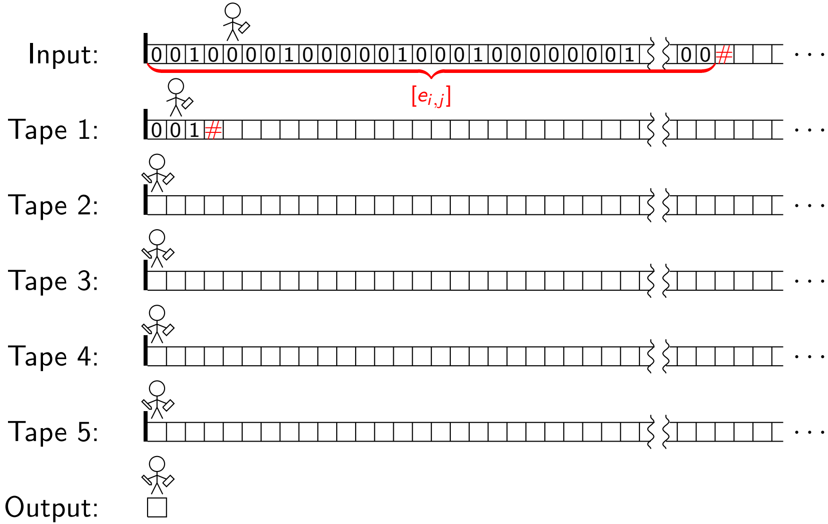

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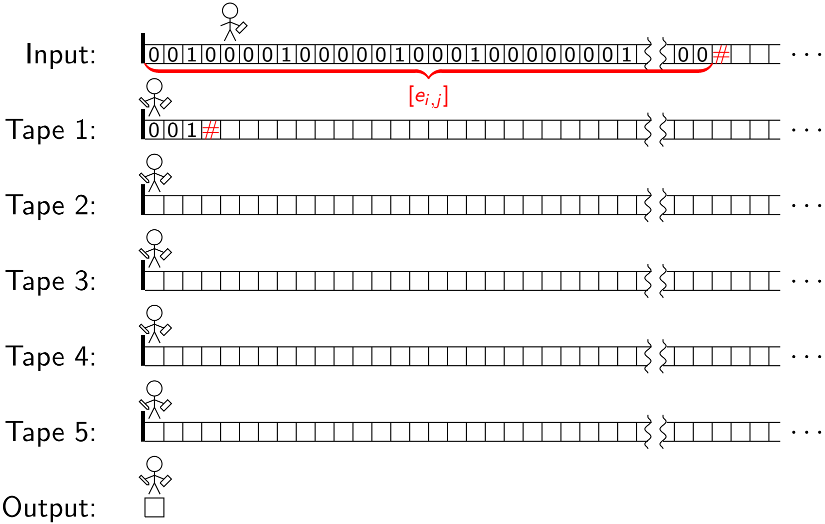

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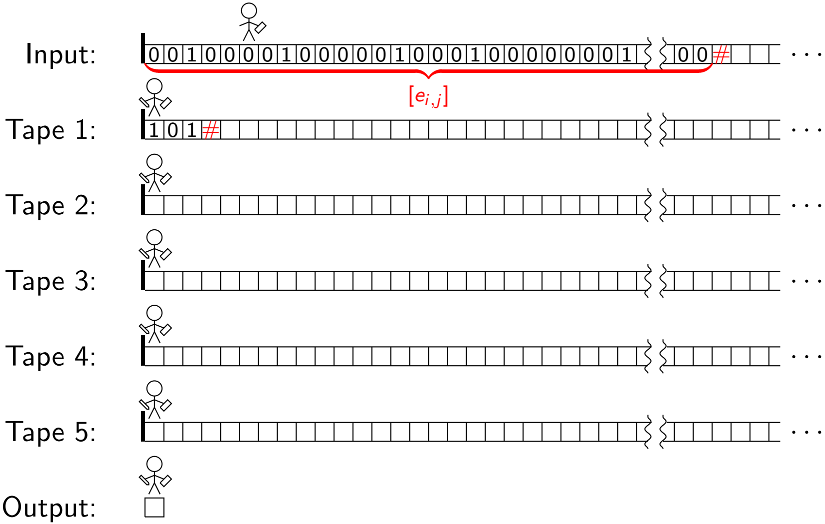

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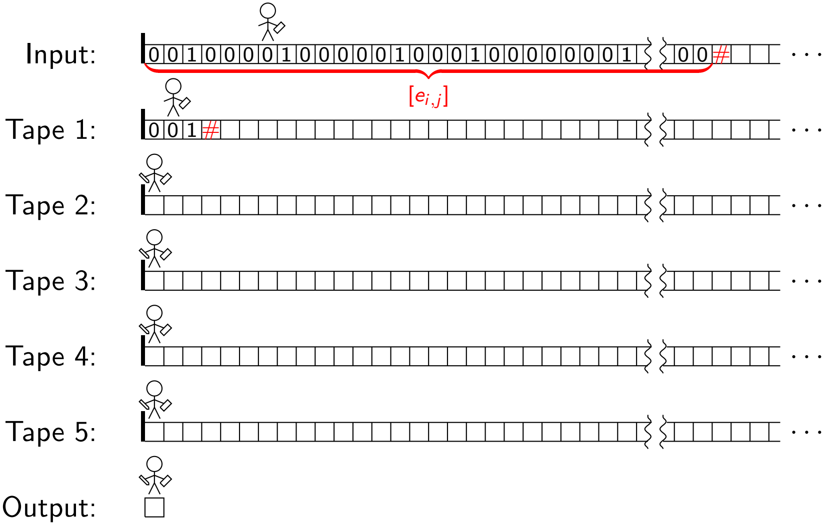

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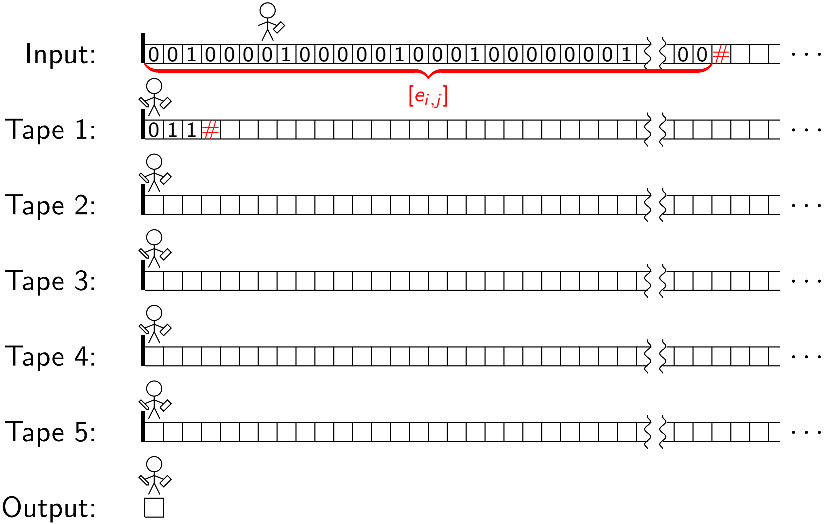
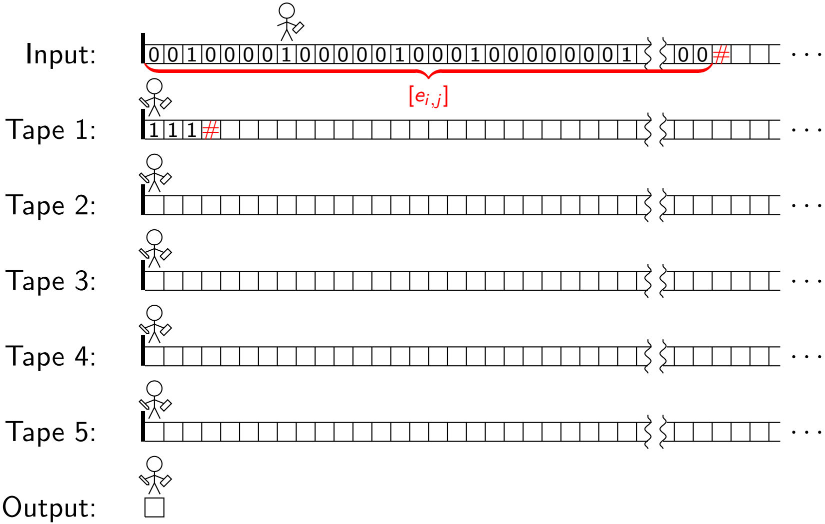
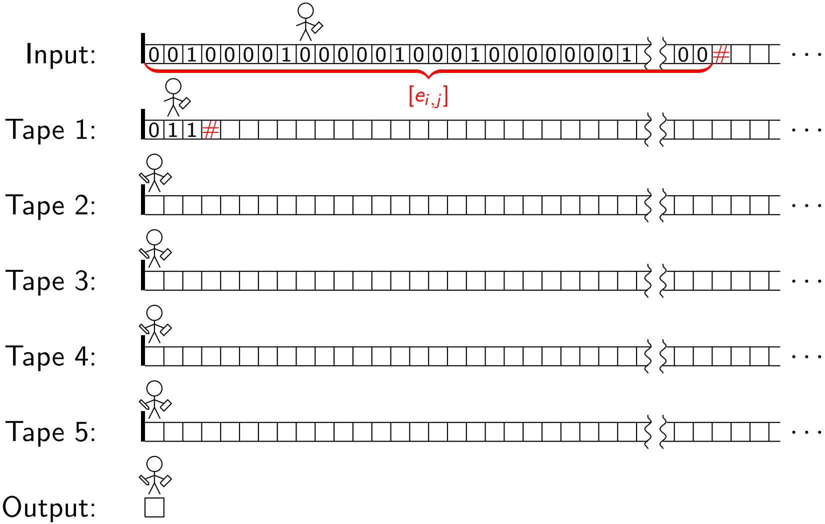




Input:
001000010000010001000000001 00#
[e_{i,j}]
Tape 1:
111#
Tape 2:
Tape 3:
Tape 4:
Tape 5:
Output:

Implementing the algorithm for *PATH*



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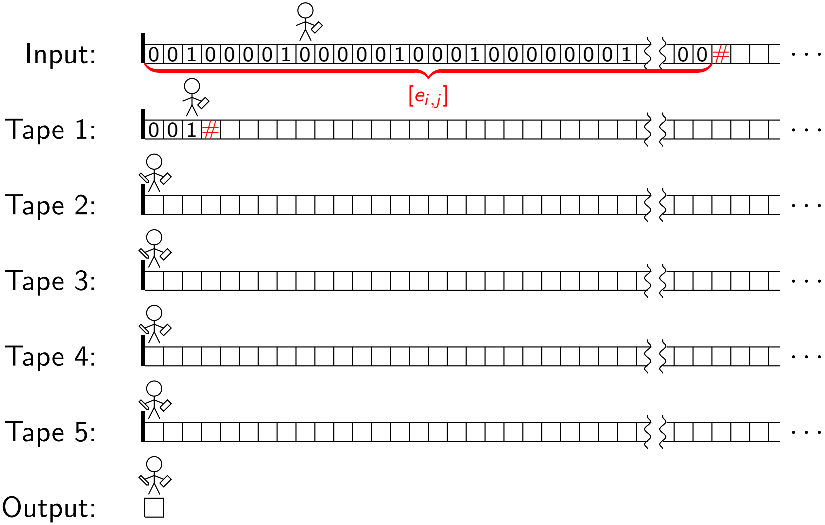

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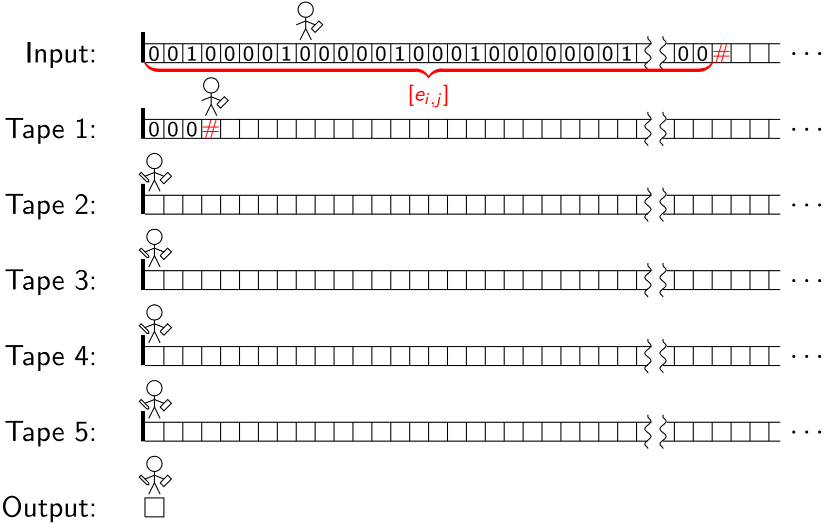

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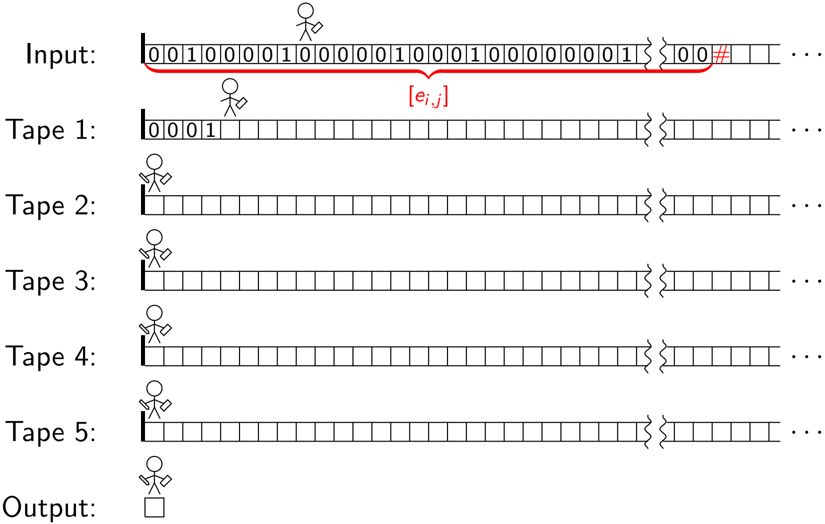

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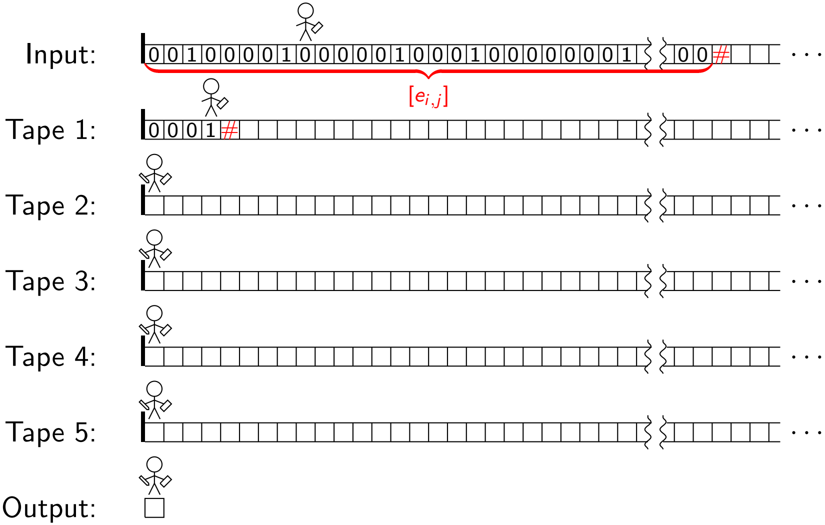

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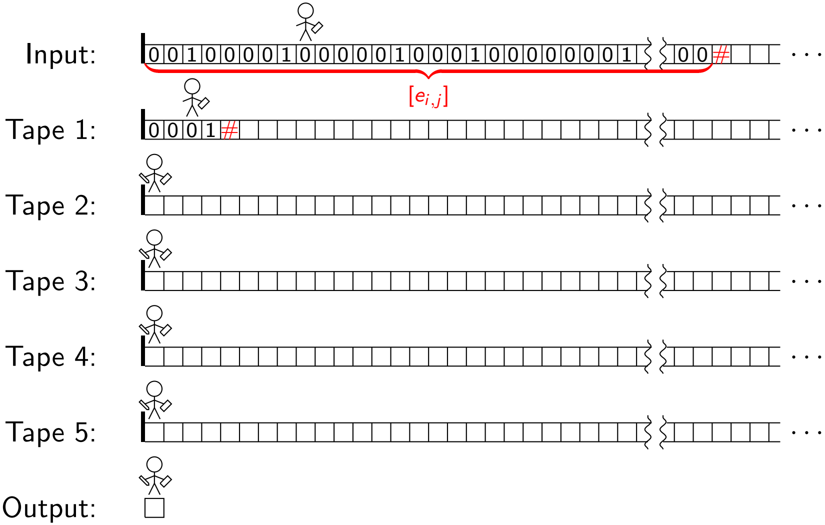

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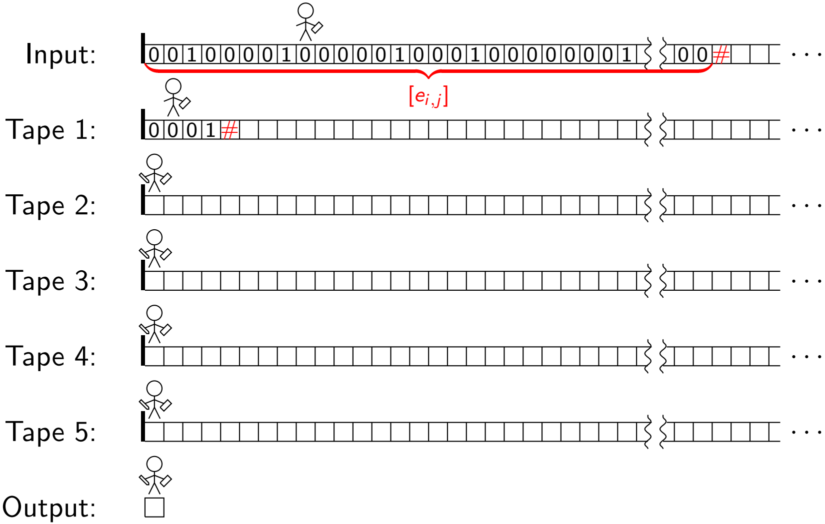

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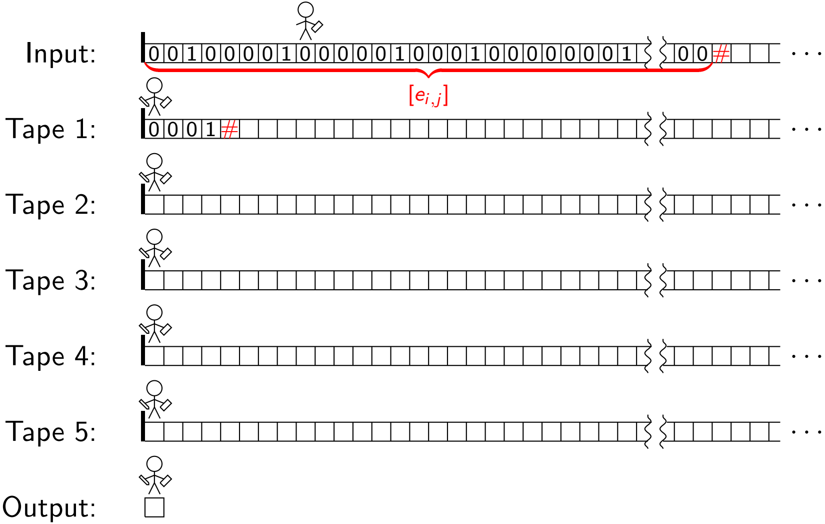
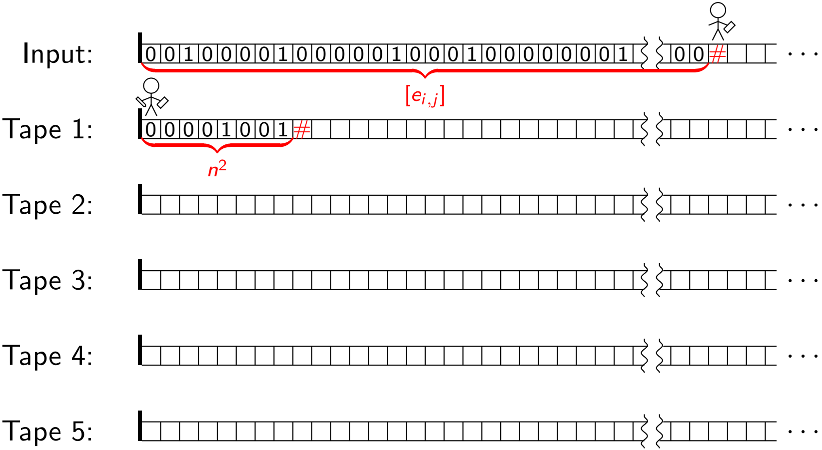


Implementing the algorithm for *PATH*

Many steps later ...

Implementing the algorithm for *PATH*



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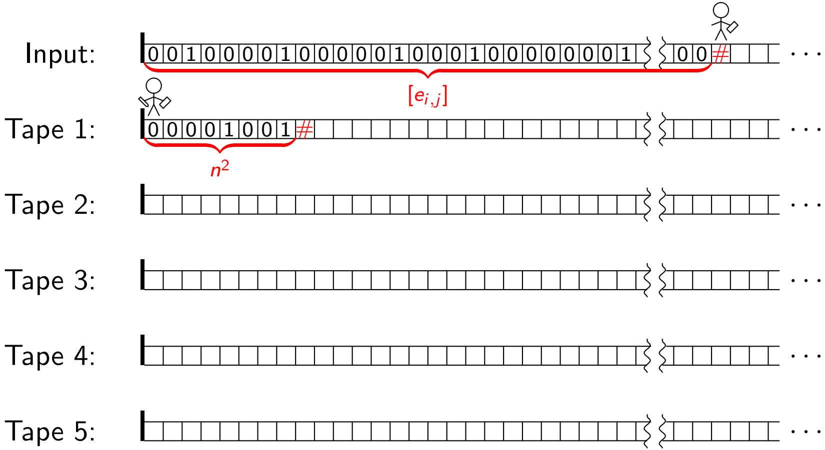
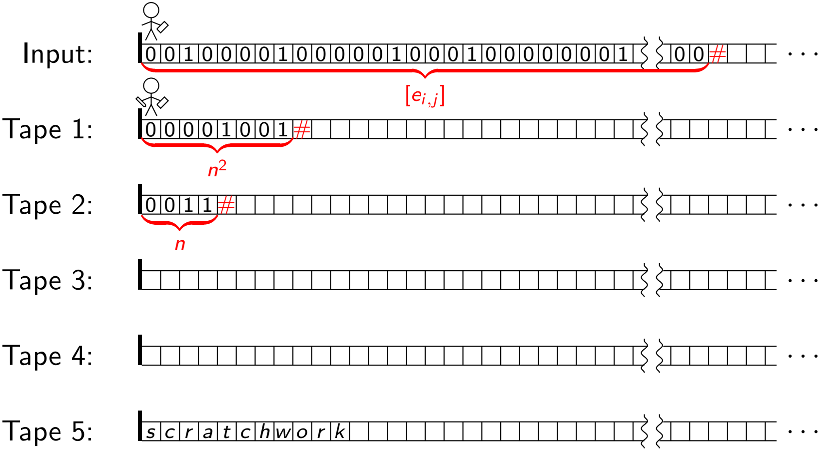


Many more steps later ...

Implementing the algorithm for *PATH*



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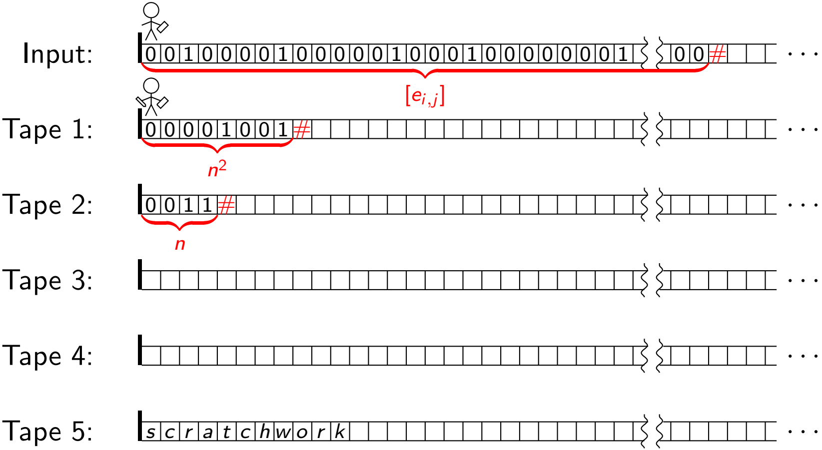
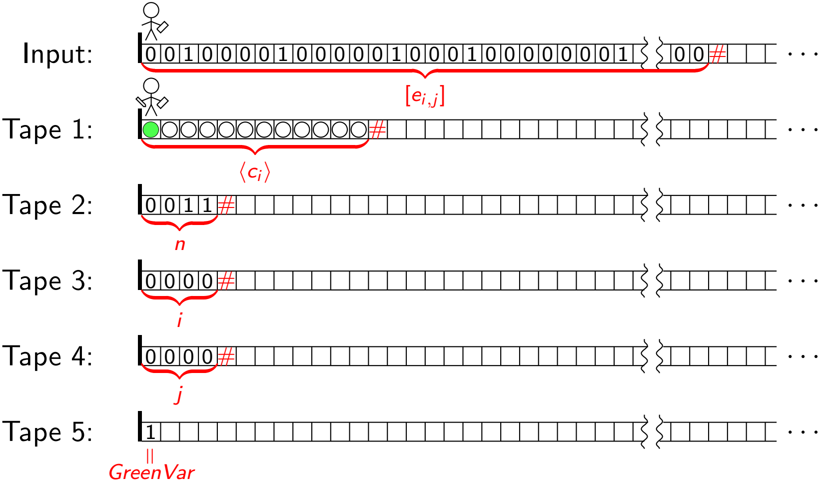


Set up variables . . .

Implementing the algorithm for *PATH*



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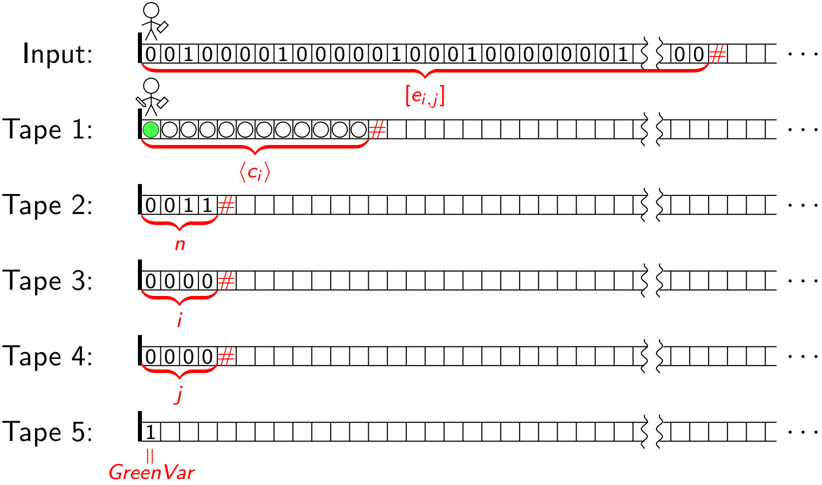
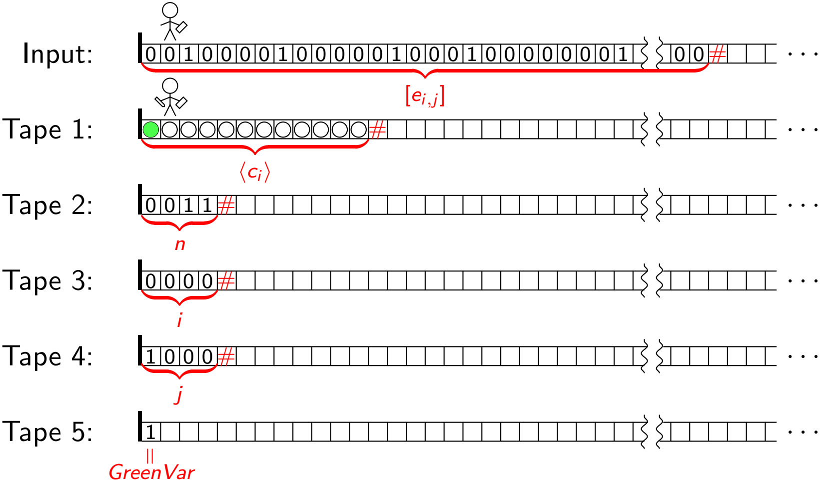


Main loop: For $i, j = 0$ to $n - 1$ . . .



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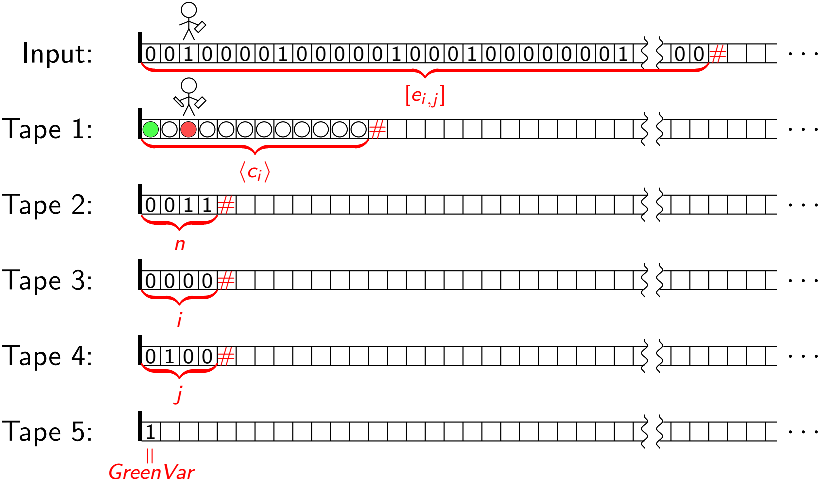

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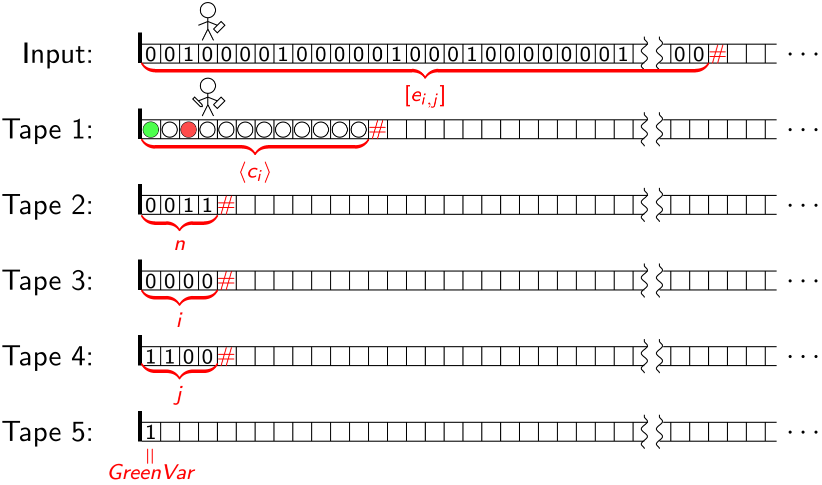

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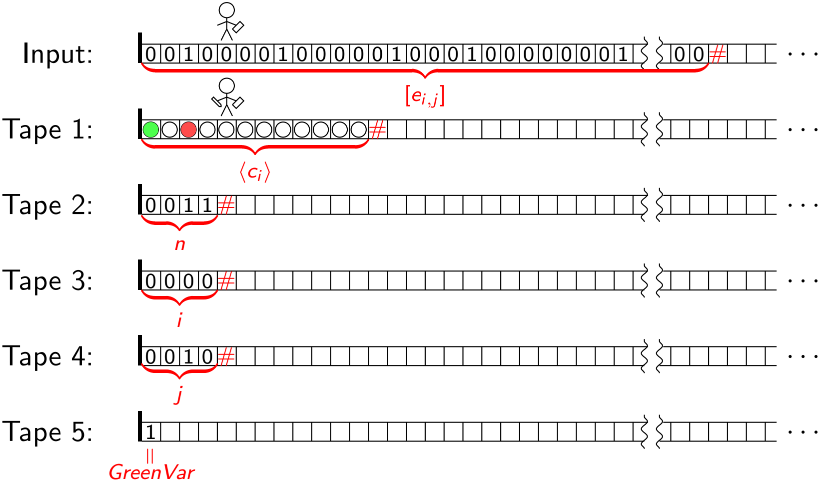

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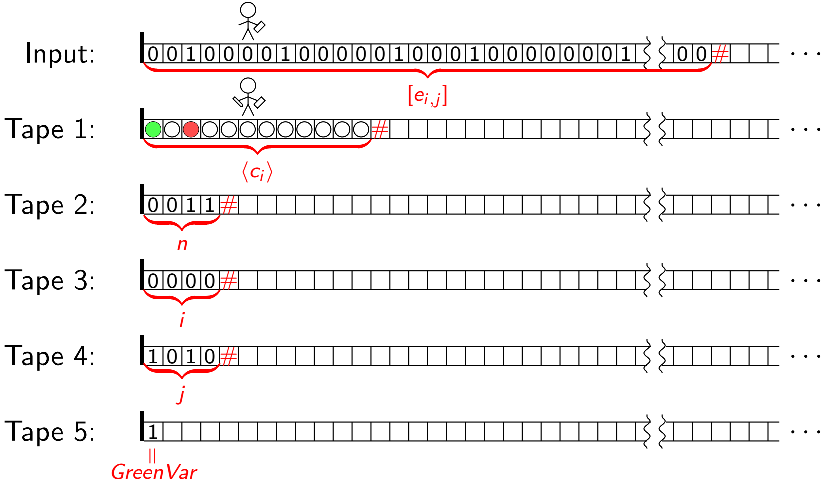

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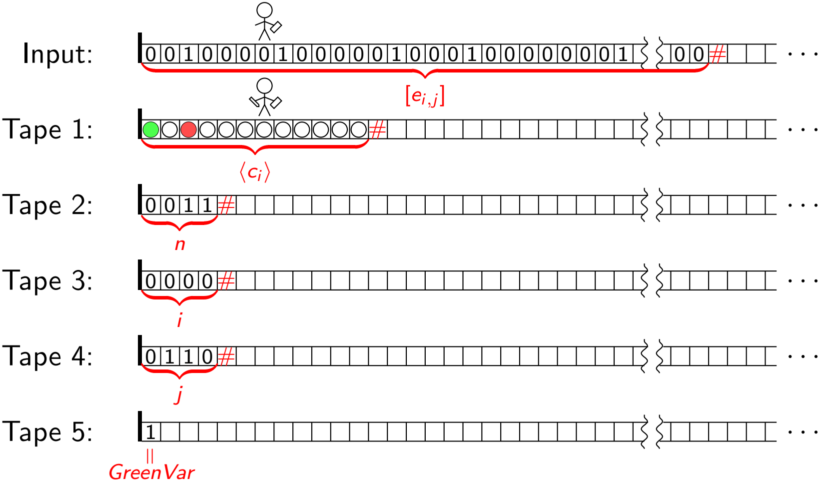

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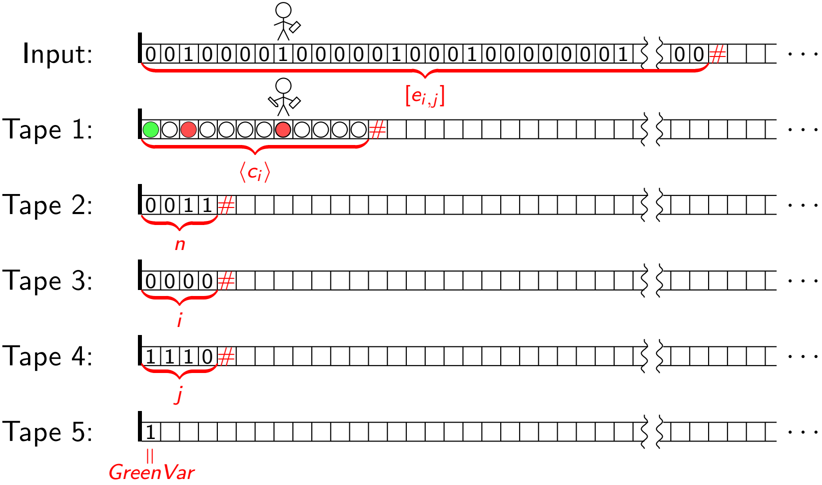

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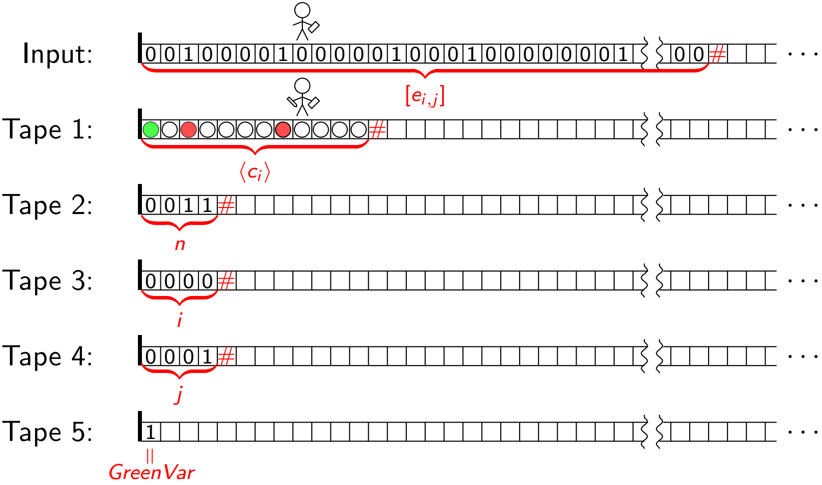

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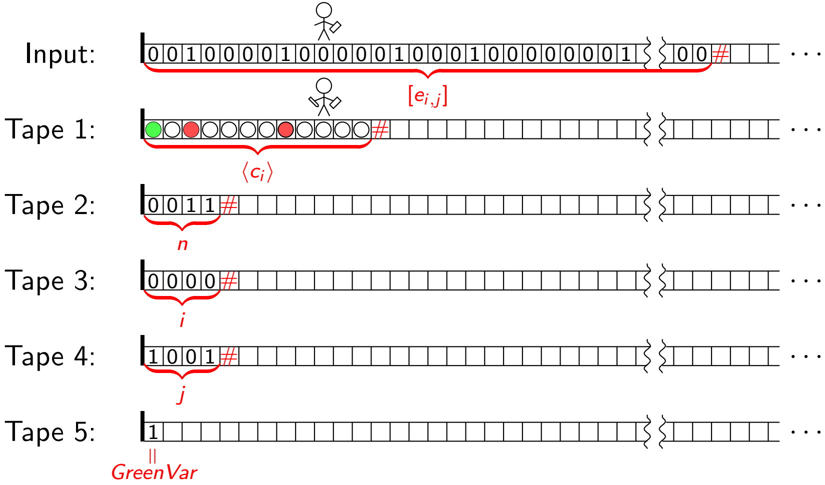

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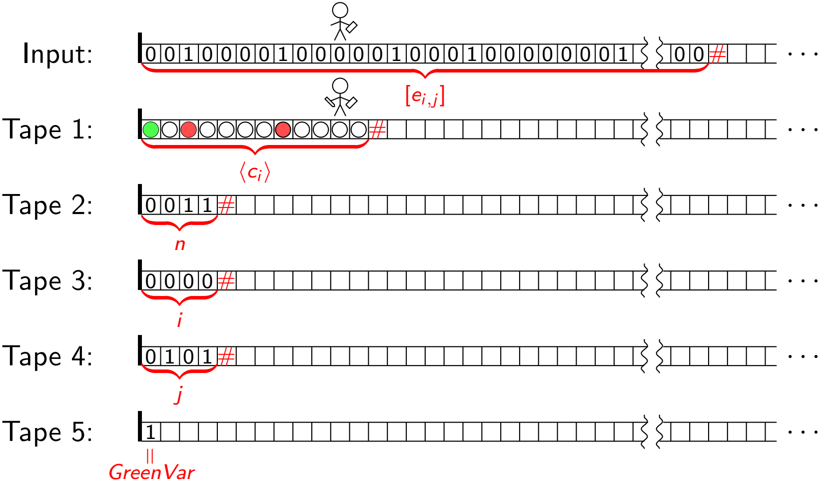

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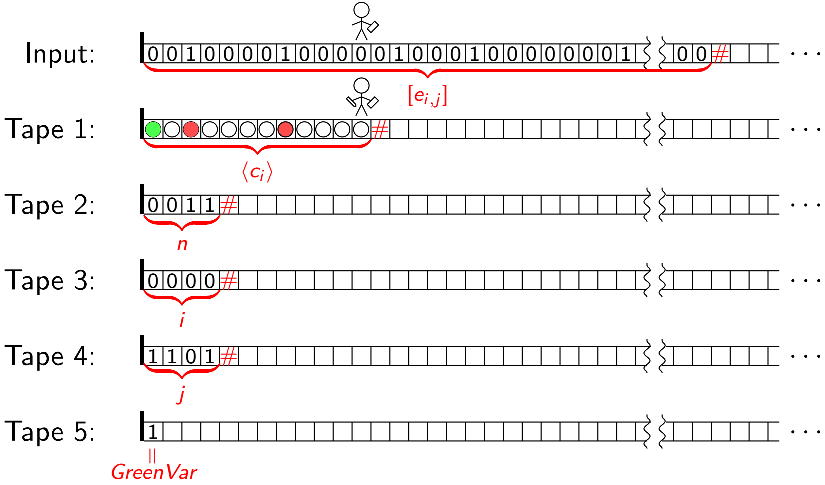

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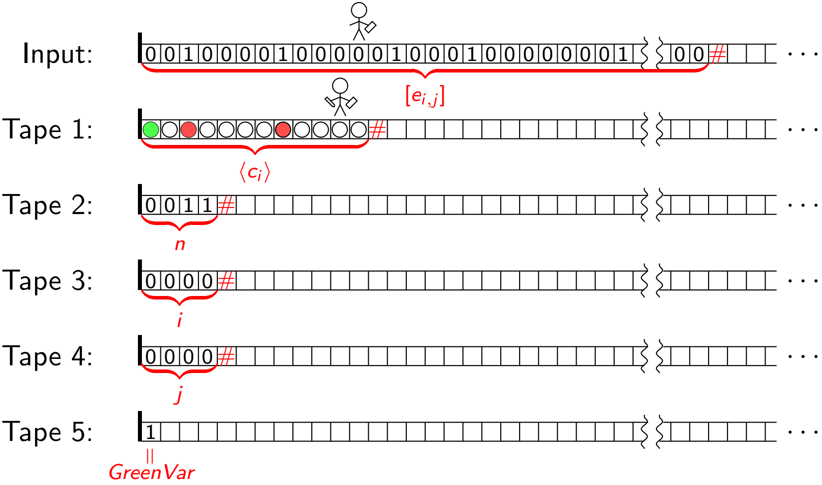

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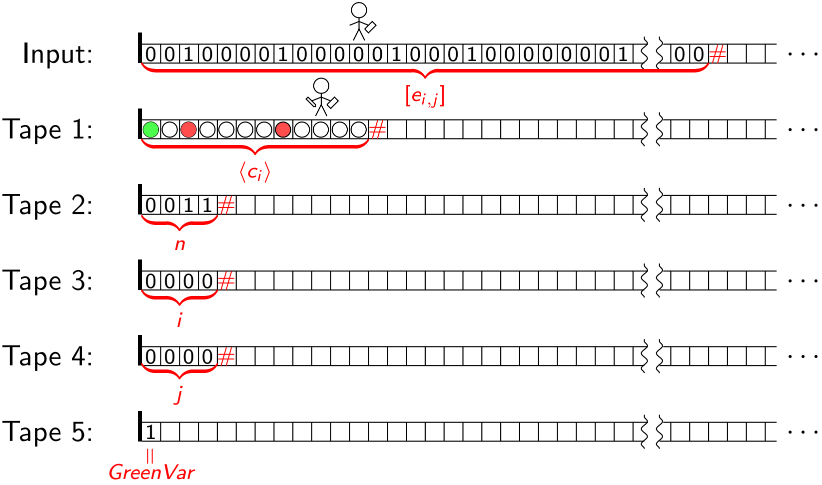

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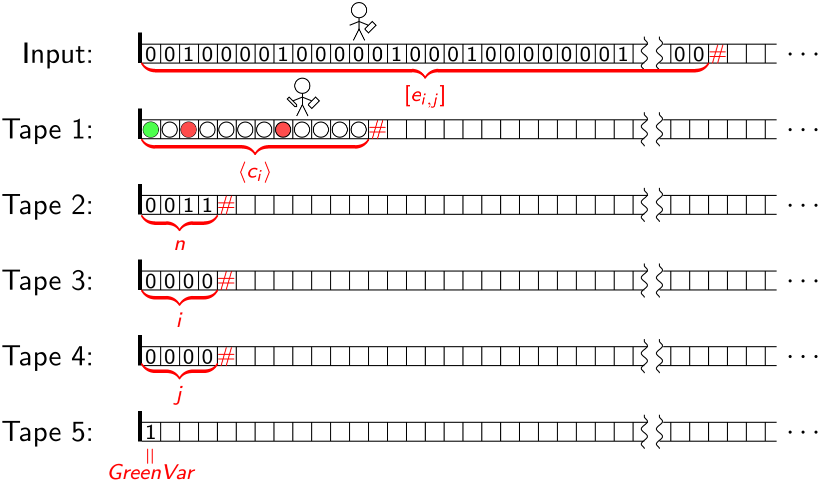

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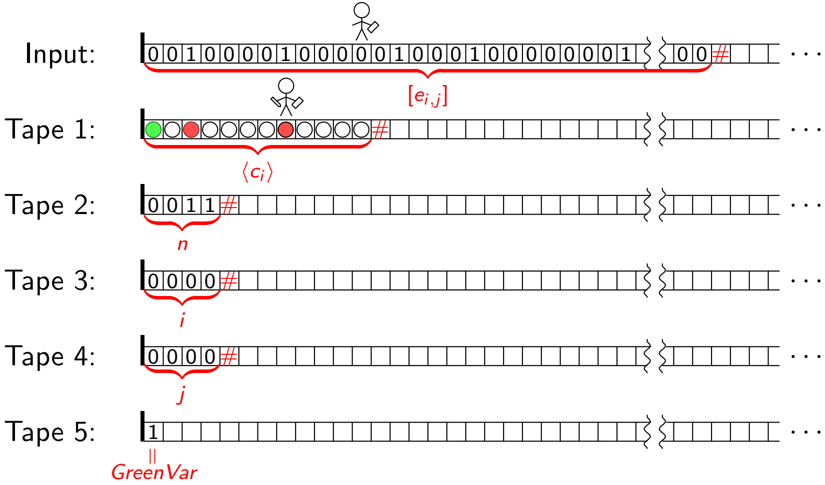

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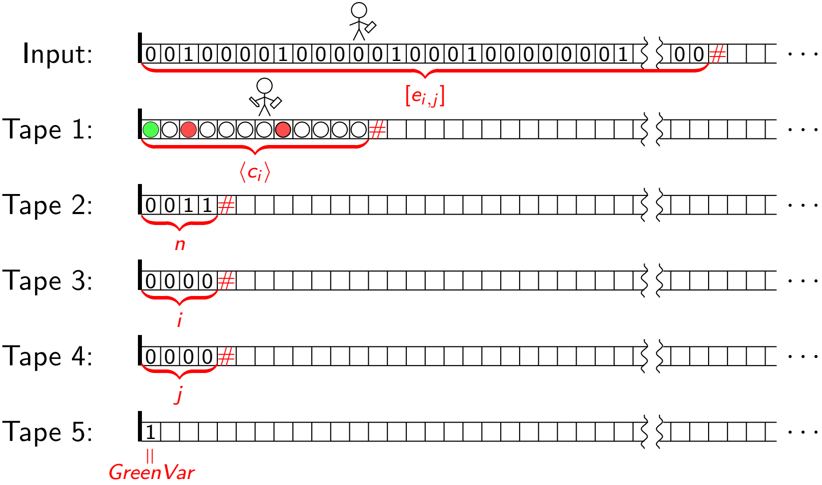

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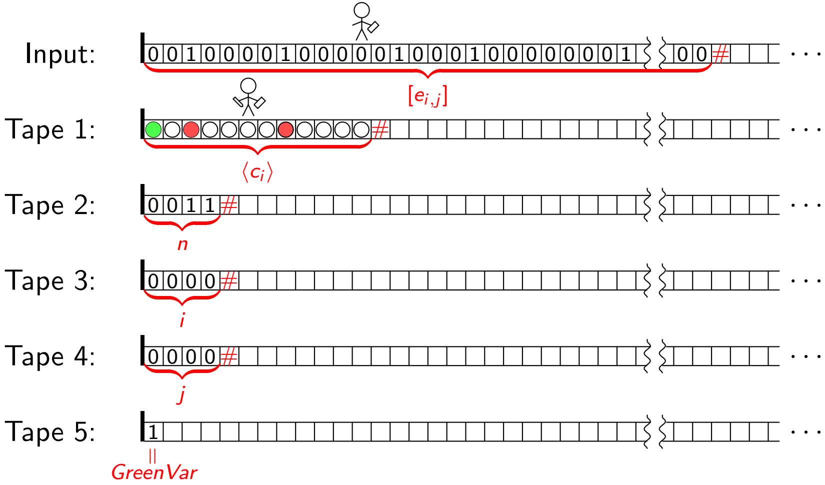

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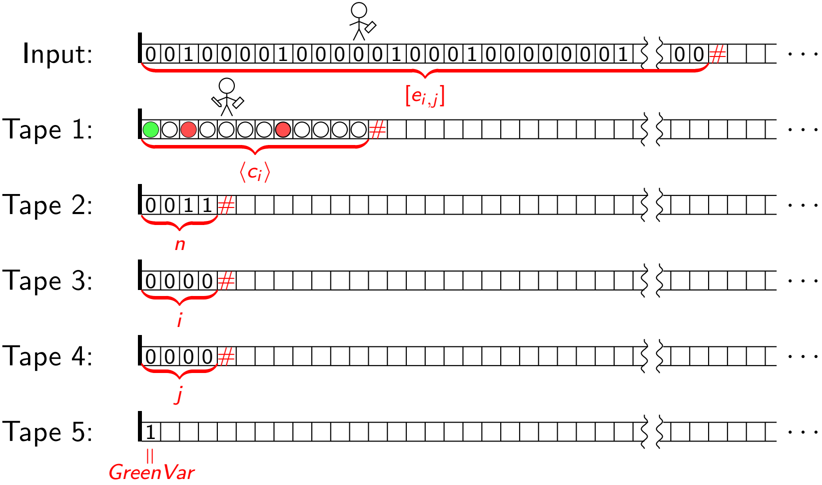

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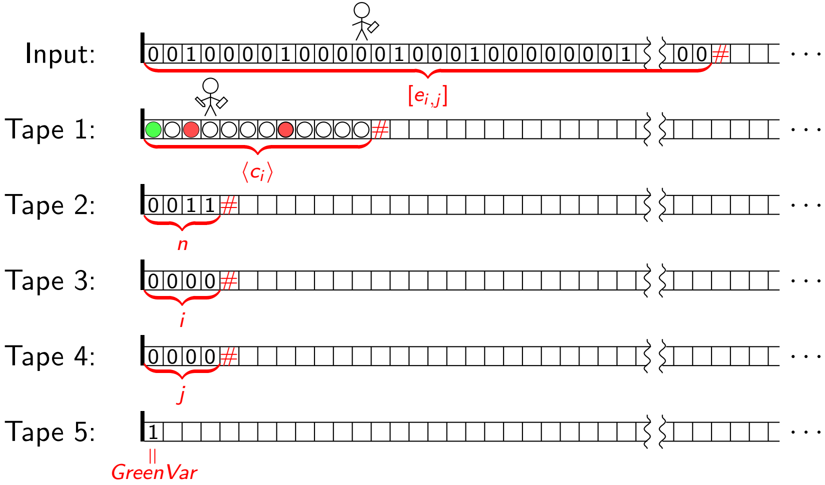

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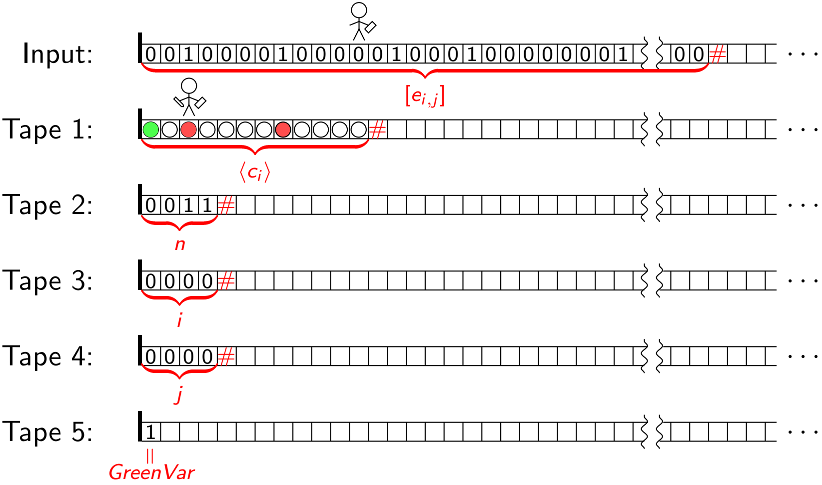

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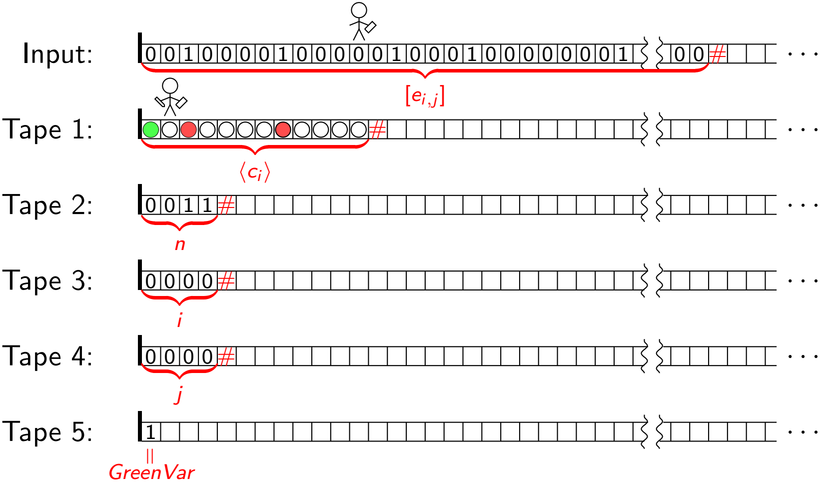

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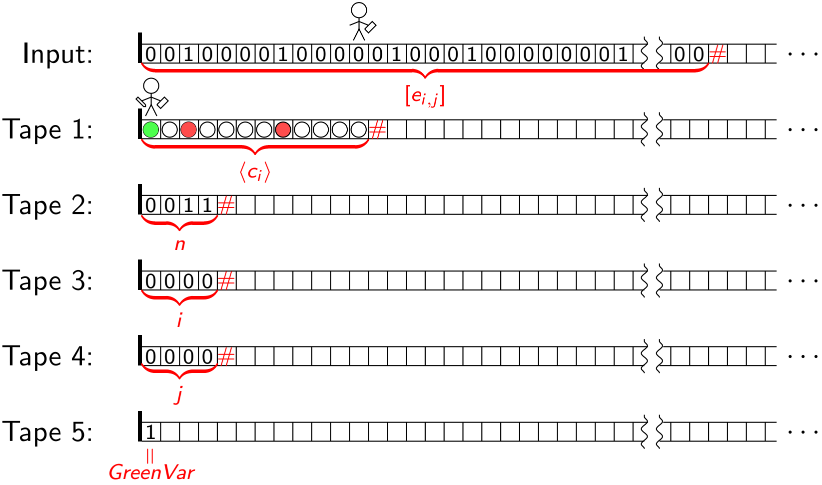

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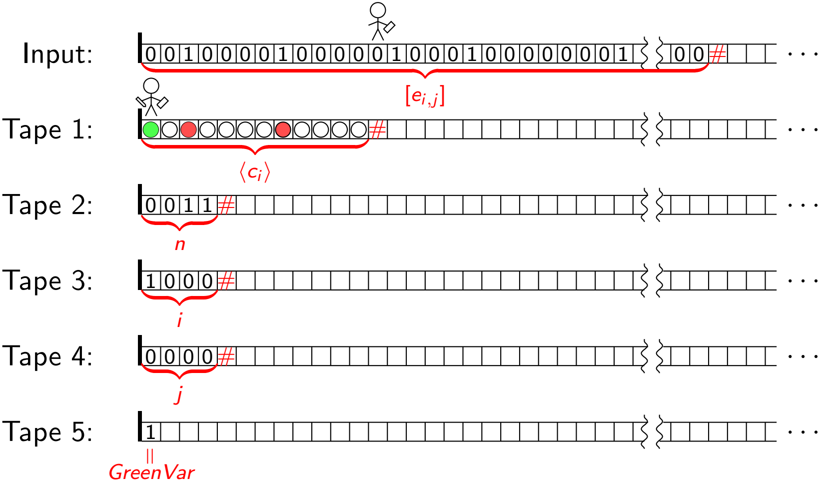

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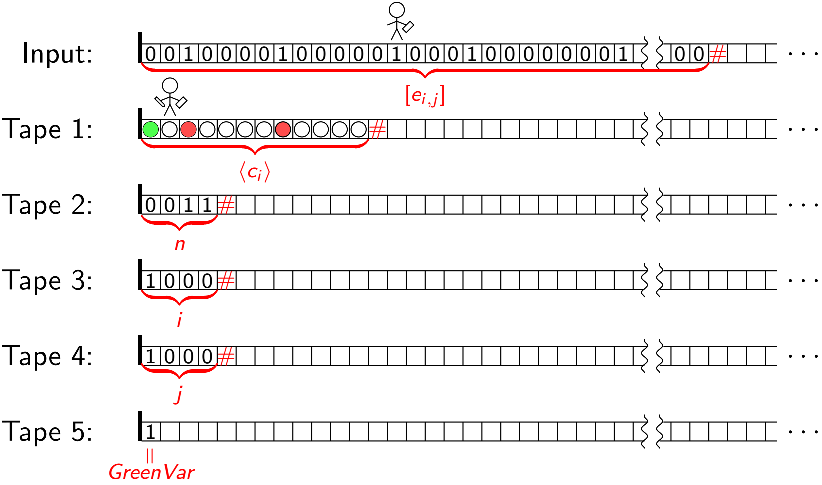

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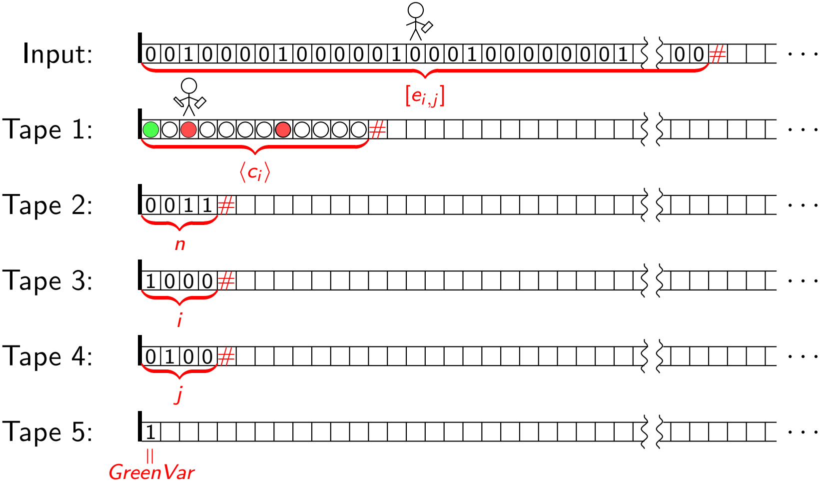

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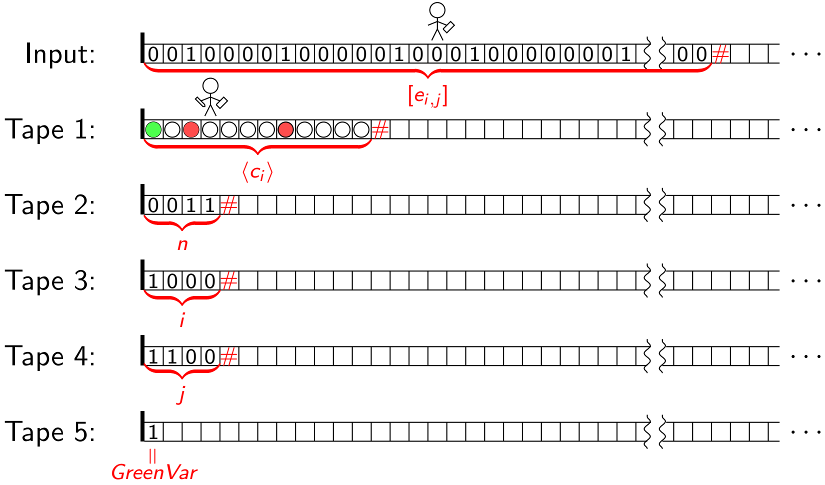

Main loop: and so on . . .

Pseudo-code revisited

Point: overhead needed to keep track of i, j, c_i, c_j.

Thus:

- While $GreenVar = yes$ do: n loops
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$ n^2 cases
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SUMMARY: on an input graph $G = (V, E)$ with $|V| = n$, our algorithm decides the answer to $PATH$ using:

Heuristics	$3n$ color changes
Pseudo-code	$O(n^3)$ operations
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Turing machine complexity

Let $f : \mathbb{N} \to \mathbb{N}$ be given.

Definition

A decision problem D (with a specified encoding of its inputs) is:

1. in $TIME(f(N))$ if there exists a Turing machine solving D in at most $O(f(N))$ steps on inputs of length N.

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Complexity of *PATH*

Recall that our Turing machine solves *PATH* on graphs with n vertices in

- Time: $O(n^4 \log n)$ steps
- Space: $O(n)$ memory cells.

Since "length N of input" $= n^2$ (when $n = |V|$), this at least proves

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Theorem

$$\begin{aligned} PATH &\in TIME(N^{2+\epsilon}) \\ PATH &\in SPACE(\sqrt{N}) \end{aligned}$$

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- Time: $O(n^4 \log n)$ steps
- Space: $O(n)$ memory cells.

Since "length N of input" $= n^2$ (when $n = |V|$), this at least proves

Theorem

$$\begin{aligned} PATH &\in TIME(N^{2+\epsilon}) \\ PATH &\in SPACE(\sqrt{N}) \end{aligned}$$

(Question: can we do better?...)

Another problem: Boolean Formula Value (*FVAL*)

INPUT:

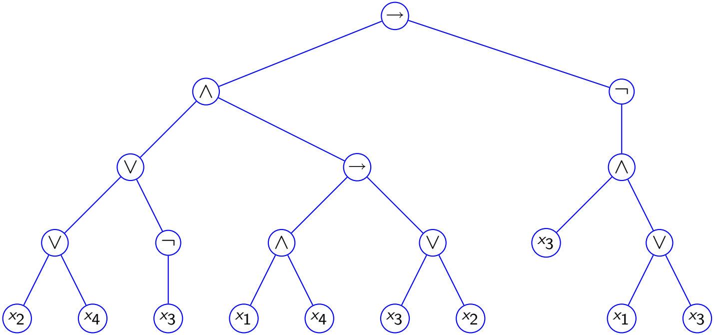
- A boolean formula φ in propositional variables $x_1, \ldots, x_n$.
- A sequence $\mathbf{c} = (c_1, \ldots, c_n) \in \{0, 1\}^n$.

QUESTION:

- Is $\varphi(\mathbf{c}) = 1$?

An algorithm for *FVAL*

$\varphi = ((((x_2 \vee x_4) \vee (\neg(x_3))) \wedge ((x_1 \wedge x_4) \rightarrow (x_3 \vee x_2))) \rightarrow (\neg(x_3 \wedge (x_1 \vee x_3))))$, $\quad \mathbf{c} = (1,0,1,1)$.



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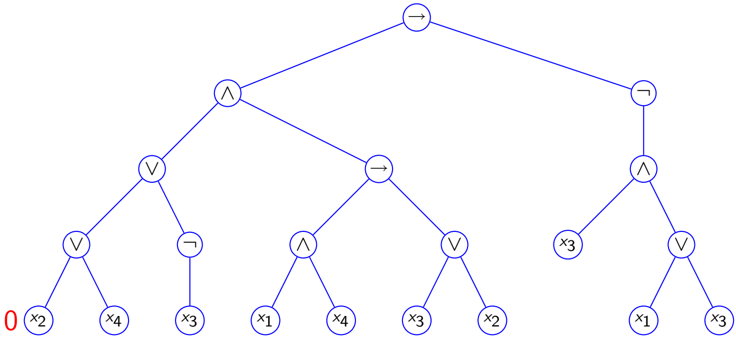

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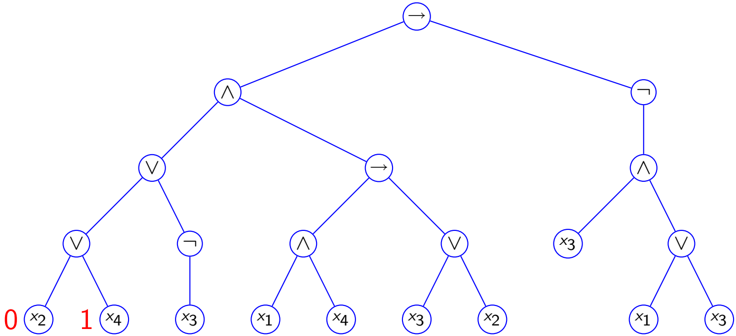

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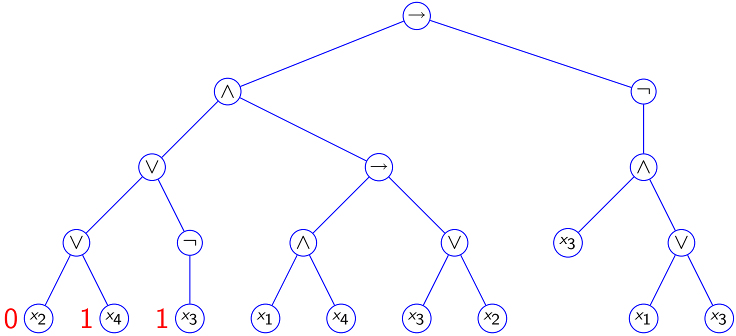

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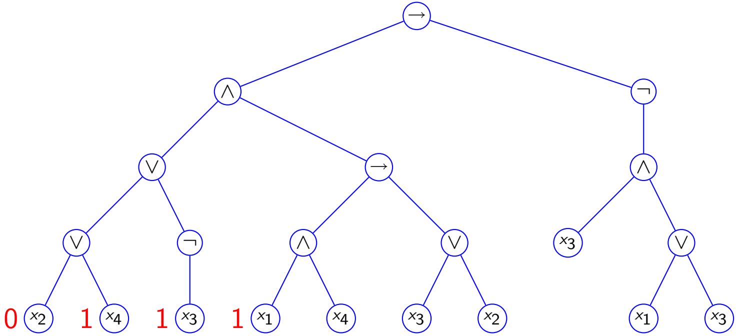

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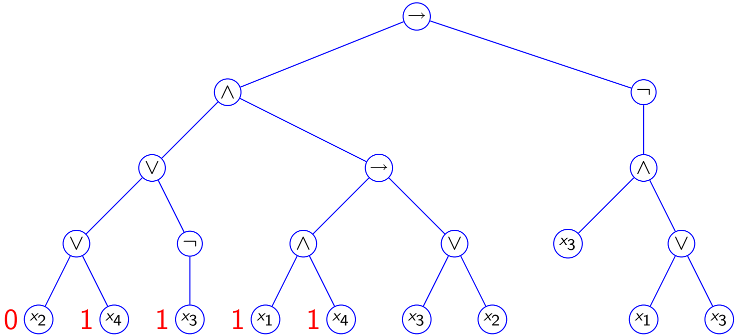

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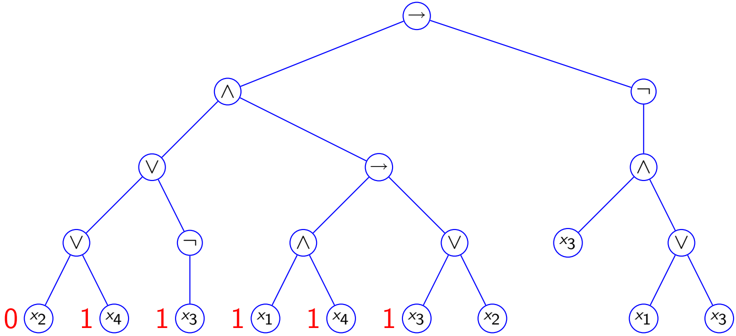

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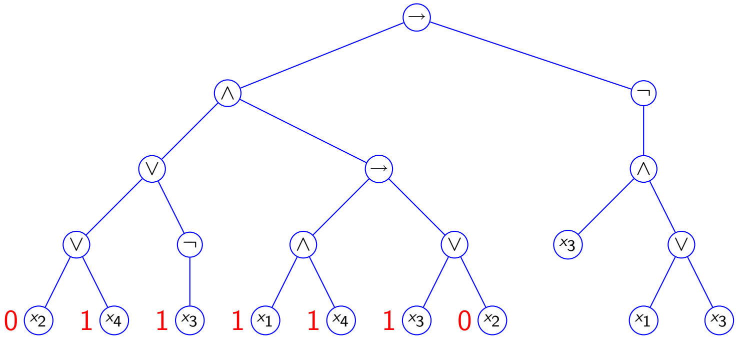

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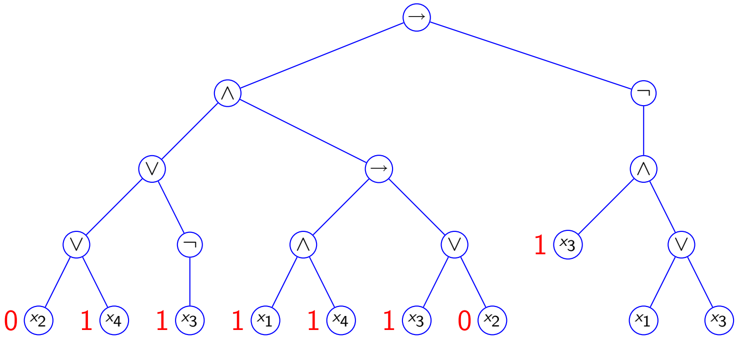

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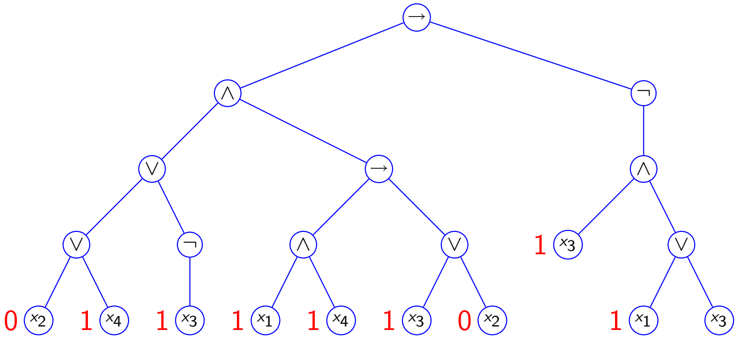

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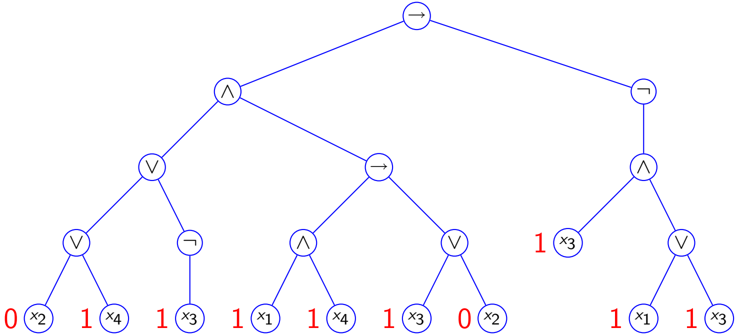

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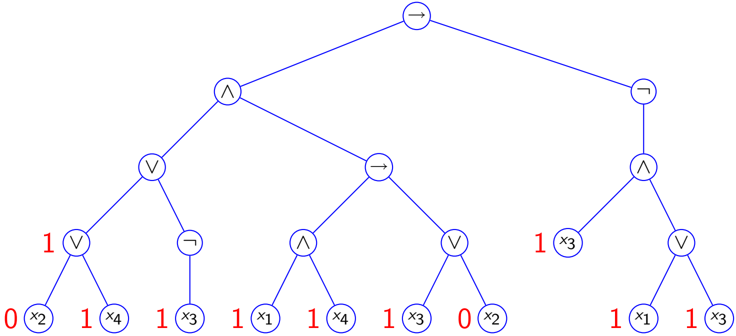

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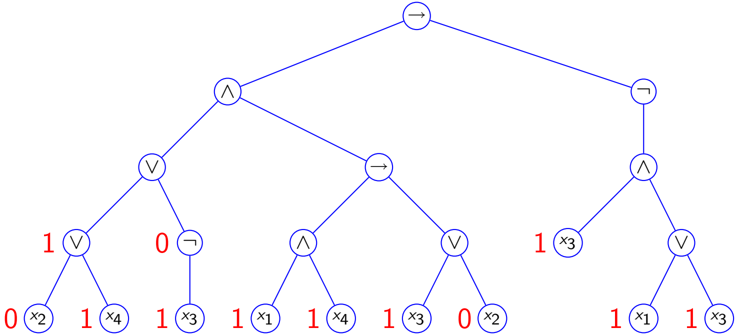

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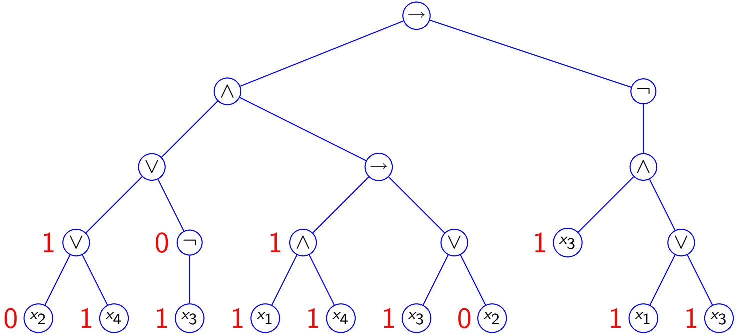

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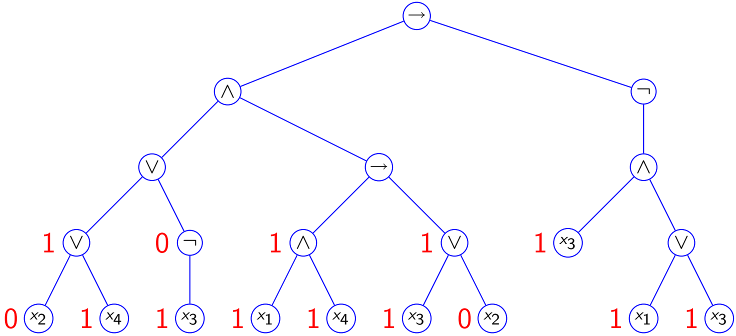

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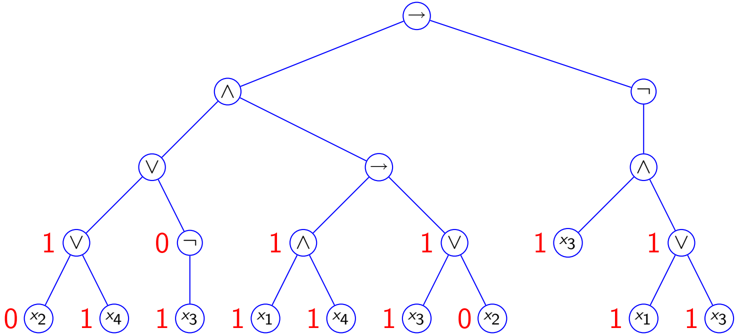

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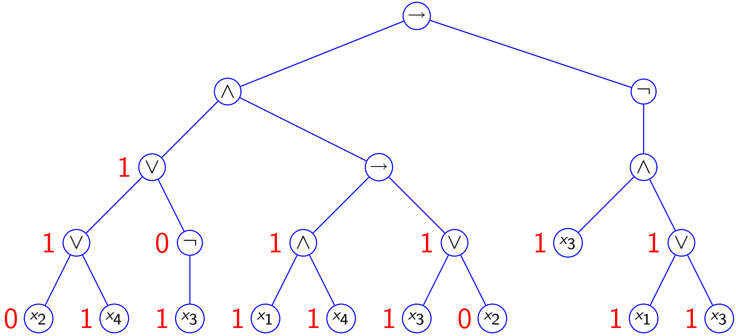

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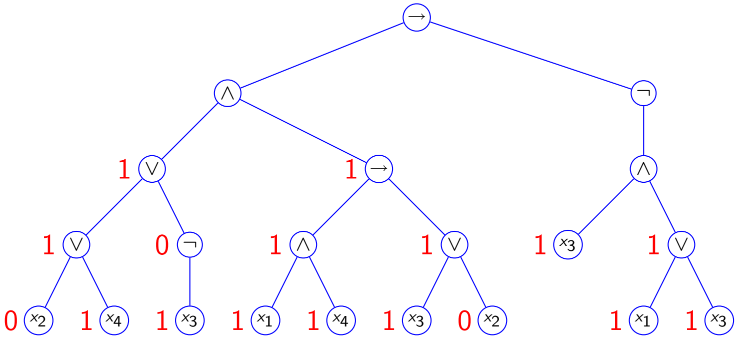

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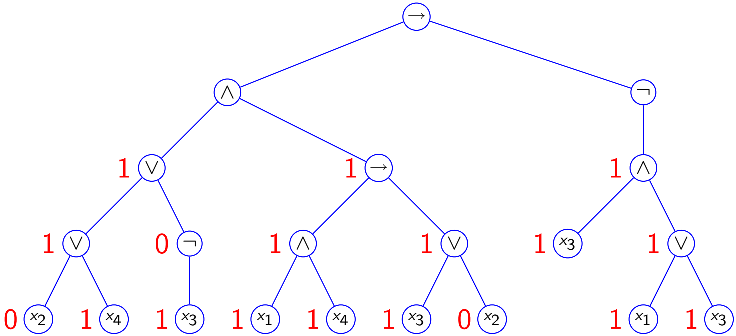

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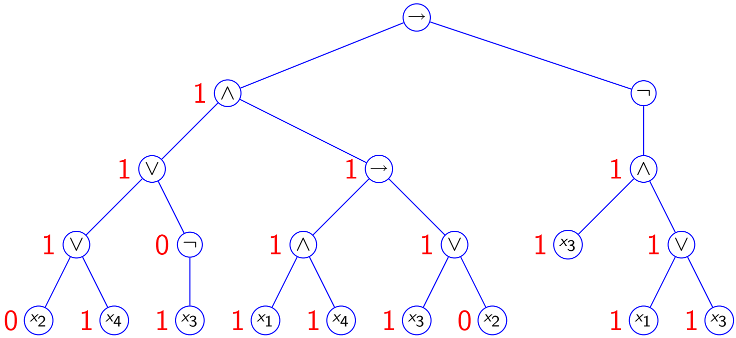

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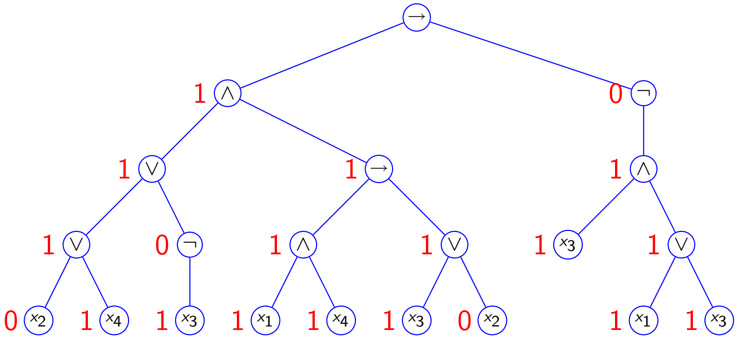

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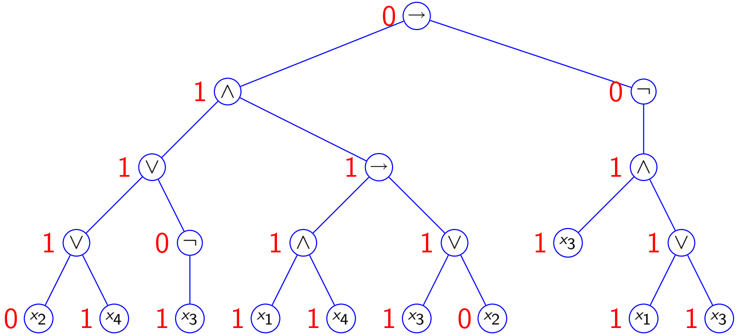

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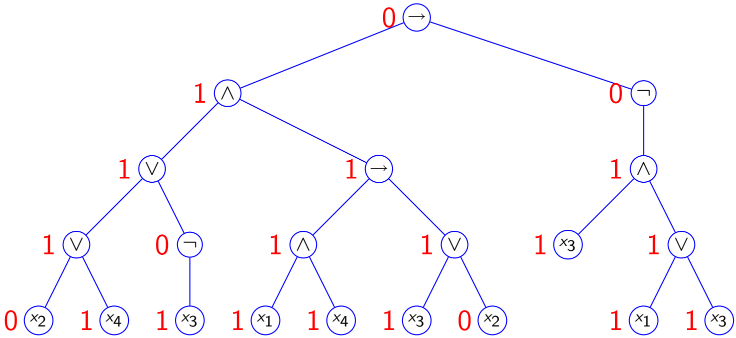

Seems to use *TIME*(N) and *SPACE*(N).

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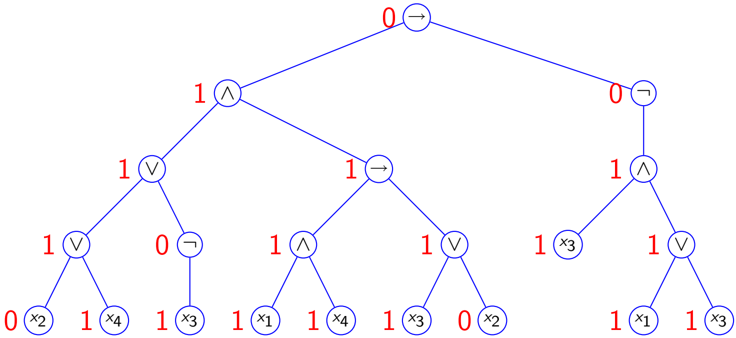

Seems to use $TIME(N)$ and $SPACE(N)$.
But space can be re-used. In this example, 3 memory bits suffice.

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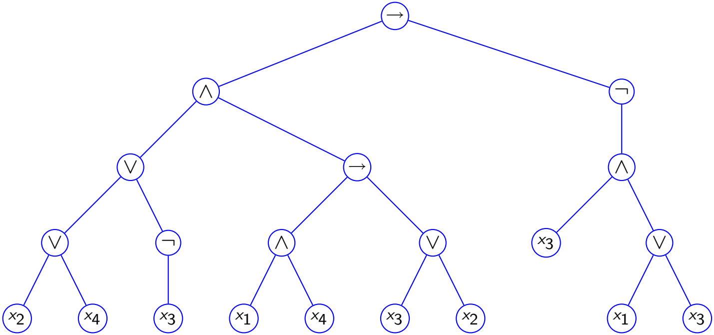

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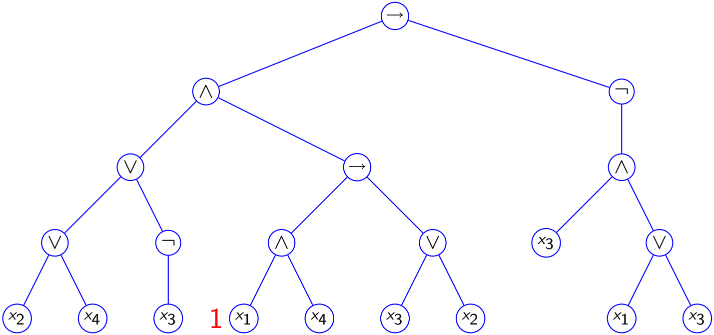

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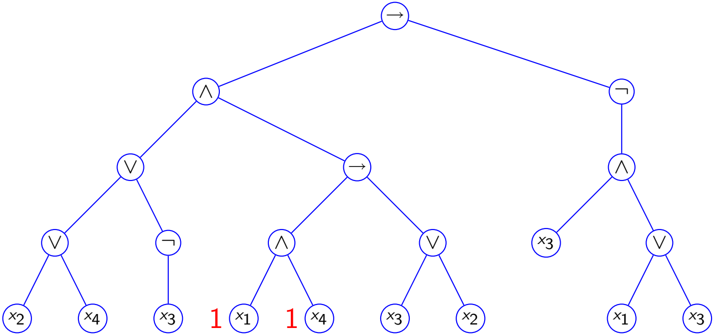

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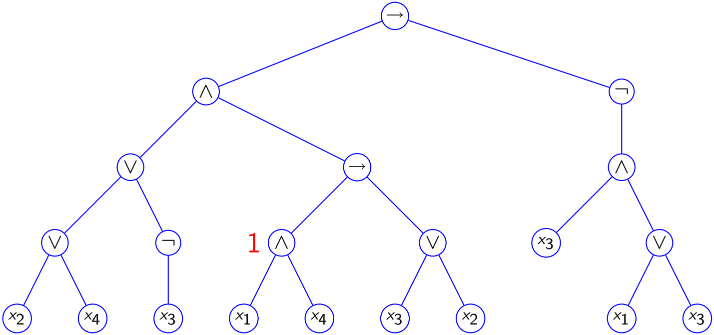

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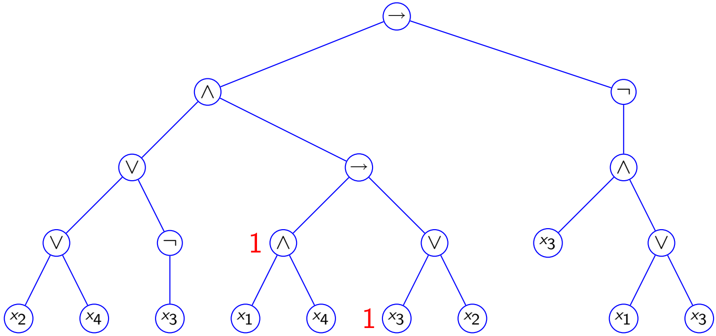

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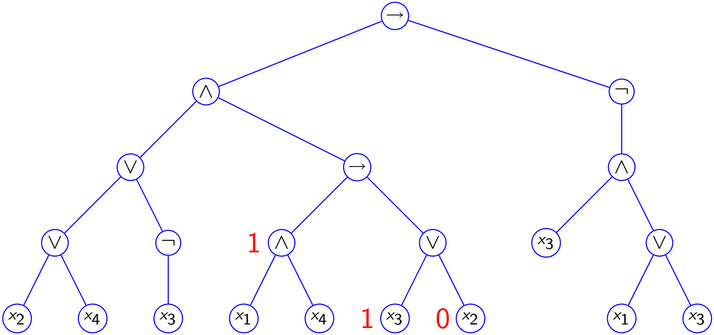

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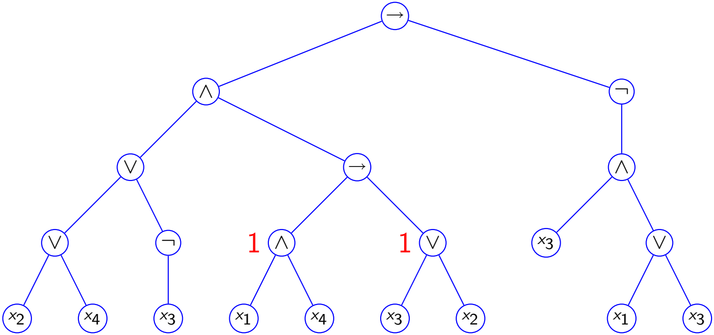

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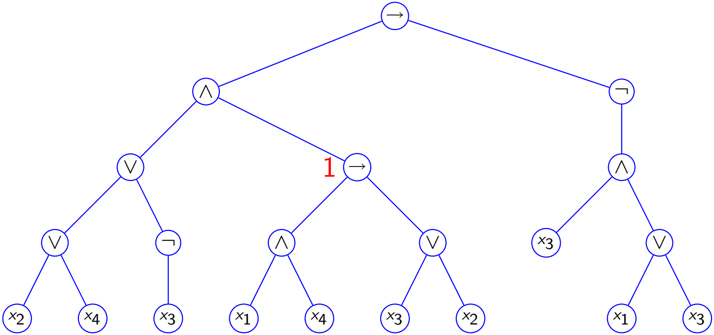

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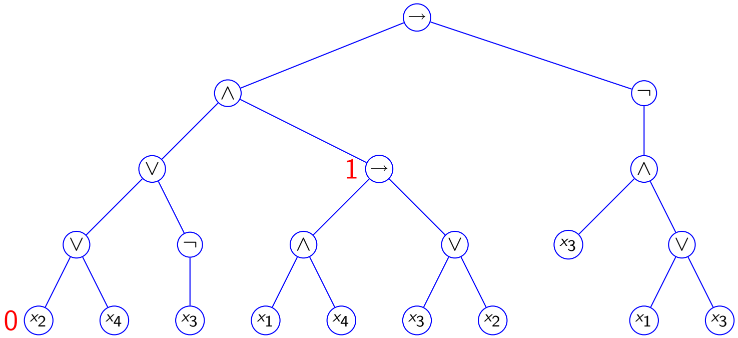

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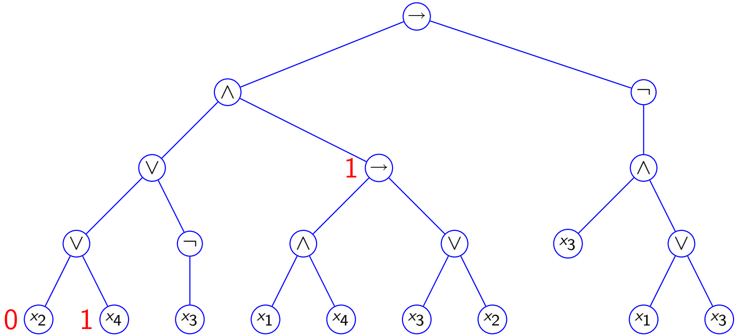

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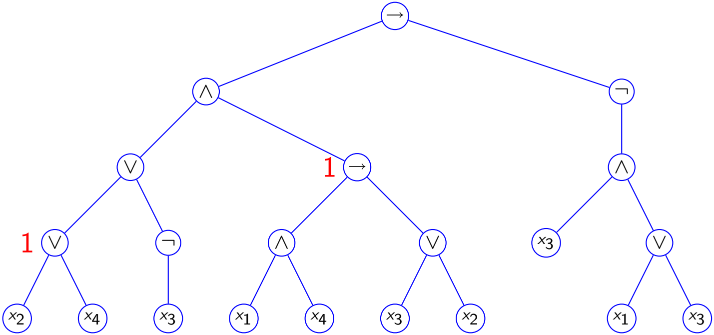

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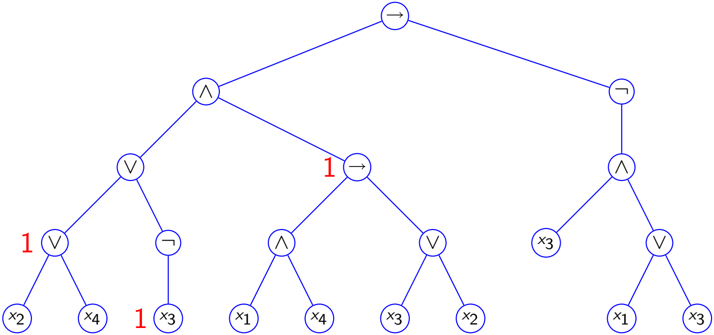

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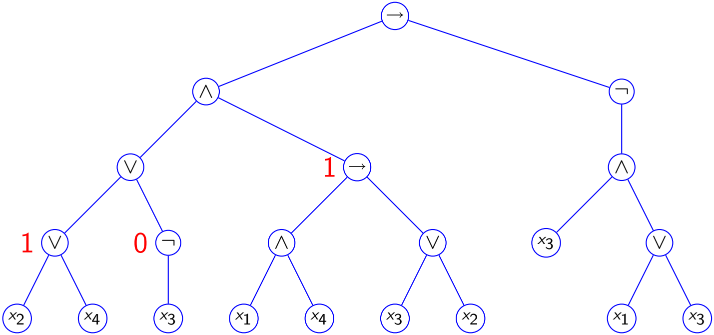

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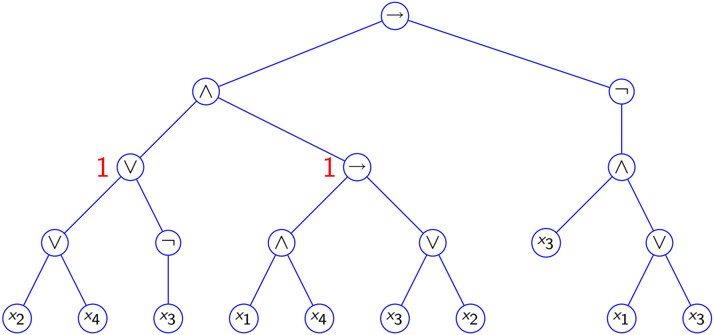

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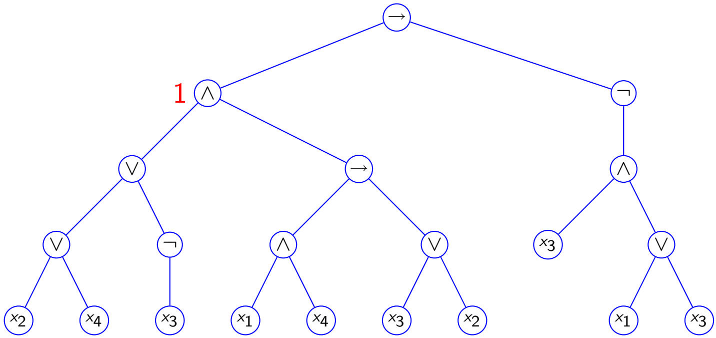

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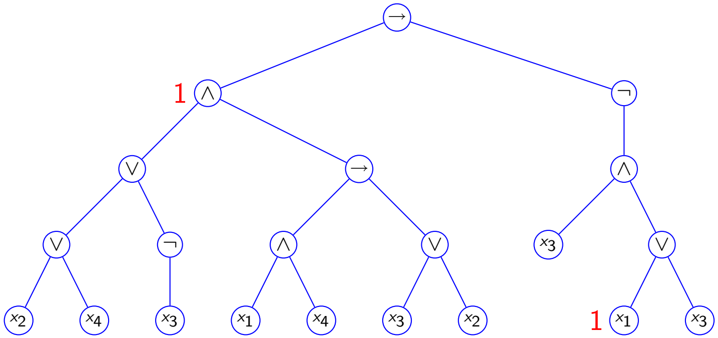

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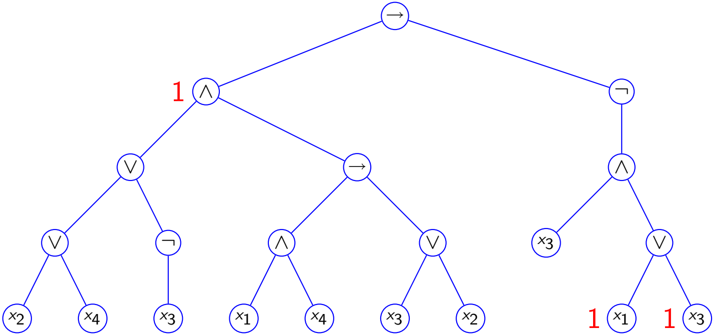

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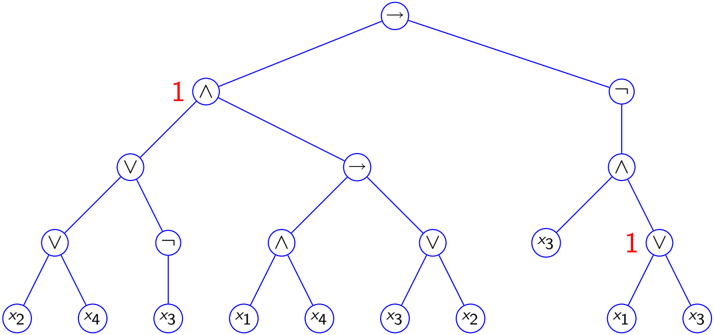

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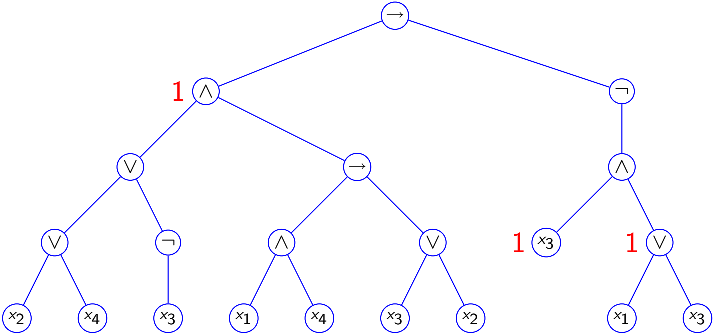

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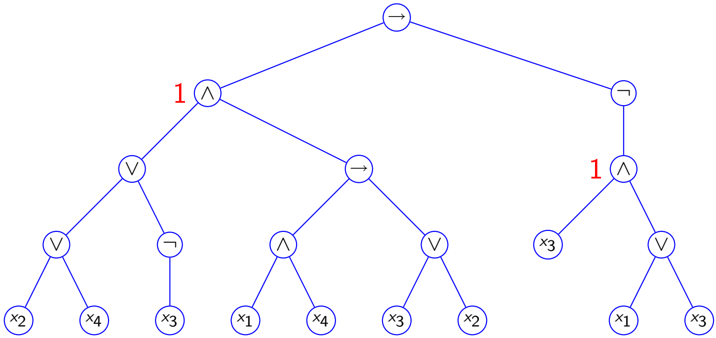

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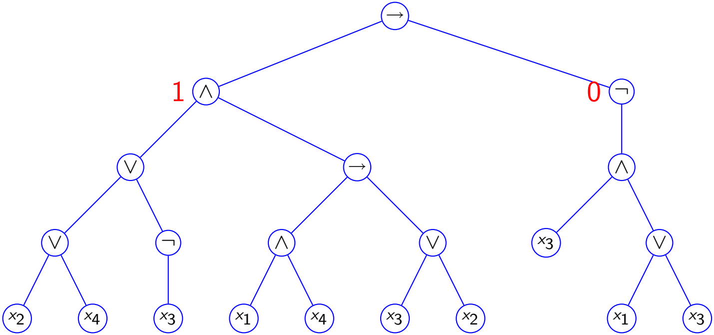

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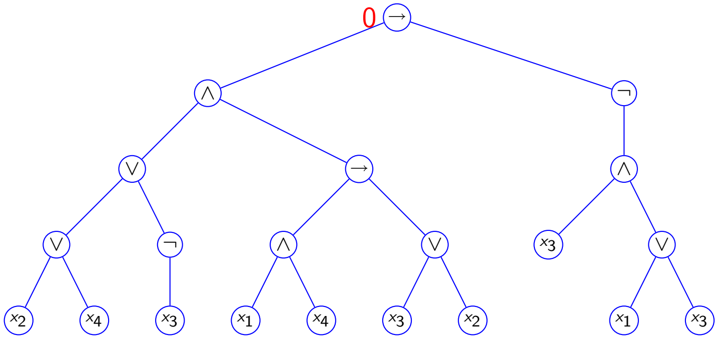

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Complexity of *FVAL*

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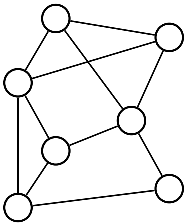
Theorem (Nancy Lynch, 1977)

$$\begin{aligned} FVAL &\in TIME(N^{2+\epsilon}) \\ FVAL &\in SPACE(\log N). \end{aligned}$$

A third problem: Graph 3-Colorability ($3COL$)

INPUT: a finite graph $G = (V, E)$.

QUESTION: Is it possible to color the vertices **red**, **green** or **blue**, so that no two adjacent vertices have the same color?



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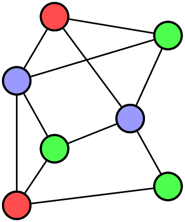

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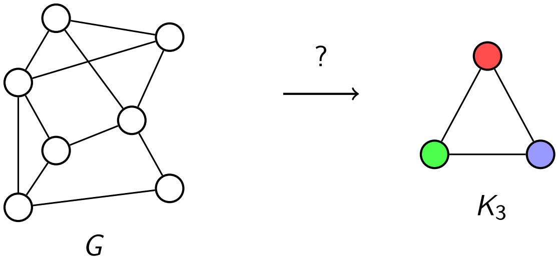
Equivalently: does there exist a homomorphism

χ :



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Brute force search algorithm:

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(Question: can we do better?...)

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In the worst case this could require as much as $O(|A^{(A^k)}|) = 2^{(|A|^k)^{1+\epsilon}}$ time and space.

I.e., exponential in the size of the input. (More on this in Lecture 3.)

Some important complexity classes

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1. $P = PTIME = \bigcup_{k=1}^{\infty} TIME(N^k) = TIME(N^{O(1)})$.
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Problems known to be in P are said to be *feasible* or *tractable*.

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Tomorrow

$$
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\end{array}
$$

In tomorrow's lecture I will:

- Introduce "nondeterministic" versions of these 4 classes.
- Introduce problems which are "hardest" for each class.