Residuated lattices

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Outline

Part I: Motivation, examples and basic theory (congruences)

Part II: Subvariety lattice (atoms and joins)

Part III: Representation, Logic, Decidability

RL examples

Boolean algebras

A Boolean algebra is a structure $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ such that (we define $\neg a = a \to 0$) [$a \to b = \neg a \to b$]

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
- for all $a, b, c \in A$,

$$a \wedge b \leq c \Leftrightarrow b \leq a \to c \text{ ($\wedge$-residuation)}$$

- for all $a \in A$, $\neg\neg a = a$ (alt. $a \vee \neg a = 1$).

Exercise. Distributivity (of $\wedge$ over $\vee$) and complementation follow from the above conditions. Also, $\wedge$-residuation can be written equationally.

Boolean algebras provide algebraic semantics for classical propositional logic.

Heyting algebras are defined without the third condition and are algebraic semantics for intuitionistic propositional logic.

Algebras of relations

Let X be a set and $Rel(X) = \mathcal{P}(X \times X)$ be the set of all binary relations on X.

For relations R, and S, we denote by

- R^- the complement and by $R^\cup$ the converse of R
- Δ is the equality/diagonal relation on X
- $R\,;S$ the relational composition of R and S
- $R\backslash S = (R\,; S^-)^-$ and $S/R = (S^-\,; R)^-$
- $R \to S = (R \cap S^-)^- = R^- \cup S$

We have

- $(Rel(X), \cap, \cup, \to, \emptyset, X^2)$ is a Boolean algebra
- $(Rel(X), ;, \Delta)$ is a monoid
- for all $R, S, T \in Rel(X)$,

$$R\,; S \subseteq T \Leftrightarrow S \subseteq R\backslash T \Leftrightarrow R \subseteq T/S.$$

Relation algebras

A Relation algebra is a structure
$\mathbf{A} = (A, \wedge, \vee, ;, \backslash, /, 0, 1, (\_)^-)$ such that $(0 = 1^-)$

- $(A, \wedge, \vee, \perp, \top, (\_)^-)$ is a Boolean algebra (we define $\perp = 1 \wedge 1^-$ and $\top = 1 \vee 1^-$),
- $(A, ;, 1)$ is a monoid
- for all $a, b, c \in A$,

$$a \,;\, b \leq c \Leftrightarrow b \leq a \backslash c \Leftrightarrow a \leq c/b \text{ (residuation)}$$

- for all $a \in A$, $\neg\neg a = a$ (we define $\neg a = a \backslash 0 = 0/a$)
- $\neg(a^-) = (\neg a)^-$ and $\neg(\neg x \,;\, \neg y) = (x^- \,;\, y^-)^-$.

ℓ-groups

A lattice-ordered group is a lattice with a compatible group structure. Alternatively, a lattice-ordered group is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid
- for all $a, b, c \in L$,

$$ab \leq c \Leftrightarrow b \leq a\backslash c \Leftrightarrow a \leq c/b.$$

- for all $a \in L$, $a \cdot a^{-1} = 1$ (we define $x^{-1} = x\backslash 1 = 1/x$).

Example. The set of real numbers under the usual order, addition and subtraction.

Powerset of a monoid

Let $\mathbf{M} = (M, \cdot, e)$ be a monoid and $X, Y \subseteq M$.
We define $X \cdot Y = \{x \cdot y : x \in X, y \in Y\}$,
$X \backslash Y = \{z \in M : X \cdot \{z\} \subseteq Y\}$,
$Y/X = \{z \in M : \{z\} \cdot X \subseteq Y\}$.

For the powerset $\mathcal{P}(M)$, we have

- $(\mathcal{P}(M), \cap, \cup)$ is a lattice
- $(\mathcal{P}(M), \cdot, \{e\})$ is a monoid
- for all $X, Y, Z \subseteq M$,

$$X \cdot Y \subseteq Z \Leftrightarrow Y \subseteq X \backslash Z \Leftrightarrow X \subseteq Z/Y.$$

Ideals of a ring

Let $\mathbf{R}$ be a ring with unit and let $\mathcal{I}(\mathbf{R})$ be the set of all (two-sided) ideals of $\mathbf{R}$.
For $I, J \in \mathcal{I}(\mathbf{R})$, we write $IJ = \{\sum_{fin} ij : i \in I, j \in J\}$
$I\backslash J = \{k : Ik \subseteq J\}$,
$J/I = \{k : kI \subseteq J\}$.

For the powerset $\mathcal{I}(\mathbf{R})$, we have

- $(\mathcal{I}(\mathbf{R}), \cap, \cup)$ is a lattice
- $(\mathcal{I}(\mathbf{R}), \cdot, R)$ is a monoid
- for all ideals I, J, K of $\mathbf{R}$,

$$I \cdot J \subseteq K \Leftrightarrow J \subseteq I\backslash K \Leftrightarrow I \subseteq K/J.$$

Residuated lattices

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \Leftrightarrow b \leq a\backslash c \Leftrightarrow a \leq c/b.$$

(We think of $x\backslash y$ and y/x as $x \to y$, when they are equal.)

A *pointed residuated lattice* an extension of a residuated lattice with a new constant 0. ($\sim x = x\backslash 0$ and $-x = 0/x$.)

A (pointed) residuated lattice is called

- commutative, if $(L, \cdot, 1)$ is commutative ($xy = yx$).
- distributive, if $(L, \wedge, \vee)$ is distibutive
- integral, if it satisfies $x \leq 1$
- contractive, if it satisfies $x \leq x^2$
- involutive, if it satisfies $\sim - x = x = - \sim x$.

Properties

1. $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$
2. $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ and $(y \wedge z)/x = (y/x) \wedge (z/x)$
3. $x/(y \vee z) = (x/y) \wedge (x/z)$ and $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$
4. $(x/y)y \leq x$ and $y(y\backslash x) \leq x$
5. $x(y/z) \leq (xy)/z$ and $(z\backslash y)x \leq z\backslash(yx)$
6. $(x/y)/z = x/(zy)$ and $z\backslash(y\backslash x) = (yz)\backslash x$
7. $x\backslash(y/z) = (x\backslash y)/z$;
8. $x/1 = x = 1\backslash x$
9. $1 \leq x/x$ and $1 \leq x\backslash x$
10. $x \leq y/(x\backslash y)$ and $x \leq (y/x)\backslash y$
11. $y/((y/x)\backslash y) = y/x$ and $(y/(x\backslash y))\backslash y = x\backslash y$
12. $x/(x\backslash x) = x$ and $(x/x)\backslash x = x$;
13. $(z/y)(y/x) \leq z/x$ and $(x\backslash y)(y\backslash z) \leq x\backslash z$

Multiplication is order preserving in both coordinates. Each division operation is order preserving in the numerator and order reversing in the denominator.

Properties (proofs)

$$
\begin{aligned}
x(y \vee z) \le w &\Leftrightarrow y \vee z \le x\backslash w \\
&\Leftrightarrow y, z \le x\backslash w \\
&\Leftrightarrow xy, xz \le w \\
&\Leftrightarrow xy \vee xz \le w
\end{aligned}
$$

$$x/y \le x/y \Rightarrow (x/y)y \le x$$

$$x(y/z)z \le xy \Rightarrow x(y/z) \le (xy)/z$$

$$[(x/y)/z](zy) \le x \Rightarrow (x/y)/z \le x/(zy)$$

$$[x/(zy)]zy \le x \Rightarrow x/(zy) \le (x/y)/z$$

$$
\begin{aligned}
w \le x\backslash(y/z) &\Leftrightarrow xw \le y/z \\
&\Leftrightarrow xwz \le y \\
&\Leftrightarrow wz \le x\backslash y \\
&\Leftrightarrow w \le (x\backslash y)/z
\end{aligned}
$$

Lattice/monoid properties

$$(z/y)(y/x)x \leq (z/y)y \leq z \Rightarrow (z/y)(y/x) \leq z/x$$

RL's satisfy no special purely lattice-theoretic or monoid-theoretic property.

Every lattice can be embedded in a (cancellative) residuated lattice.

Every monoid can be embedded in a (distributive) residuated lattice.

Linguistics (verbs)

We want to assign (a limited number of) linquistic types to English words, as well as to phrases, in such a way that we will be able to tell if a given phrase is a (syntacticly correct) sentence.

We will use n for 'noun phrase' and s for 'sentence'.

For phrases we use the rule: if $A : a$ and $B : b$, then $AB : ab$.

We write $C : a\backslash b$ if $A : a$ implies $AC : b$, for all A.

Likewise, $C : b/a$ if $A : a$ implies $CA : b$, for all A.

We assign type n to 'John.' Clearly, 'plays' has type $n\backslash s$, as all *intransitive* verbs.

John	plays	
n	$n\backslash s$	$n(n\backslash s) \leq s$

Some words may have more than one type. We write $a \leq b$ if every word with type a has also type b.

Linguistics (adverbs)

(John	plays)	here	$[n(n\backslash s)](s\backslash s) \leq s(s\backslash s) \leq s$
n	$n\backslash s$	$s\backslash s$	
John	(plays	here)	$s\backslash s \leq (n\backslash s)\backslash(n\backslash s)$
n	$n\backslash s$	$(n\backslash s)\backslash(n\backslash s)$	

Note that 'plays' is also a *transitive* verb, so it has type $(n\backslash s)/n$.

John	(plays	football)	$[n((n\backslash s)/n)]n \leq s$
n	$(n\backslash s)/n$	n	
(John	plays)	football	$(n\backslash s)/n \leq n\backslash(s/n)$
n	$n\backslash(s/n)$	n	$n[(n\backslash(s/n))n] \leq s$

Also, for 'John *definitely* plays football', note that we need to have $s\backslash s \leq (n\backslash s)/(n\backslash s)$.

Q: Can we decide (in)equations in residuated lattices?

Congruences

Congruences G, B

Definition. A *congruence* on an algebra $\mathbf{A}$ is an equivalence relation on A that is compatible with the operations of $\mathbf{A}$. (Alt.the kernel of a homomorphism out of $\mathbf{A}$.)

Congruences in groups correspond to normal subgroups.

Given a congruence θ on a group $\mathbf{G}$, the congruence class $[1]_\theta$ of 1 is a normal subgroup.

Given a normal subgroup N of a group $\mathbf{G}$, the relation θ_N is a congruence, where $(a,b) \in \theta_N$ iff $a \backslash b \in N$ iff $\{a\backslash b, b\backslash a\} \subseteq N$.

Congruences in Boolean algebras correspond to filters.

Given a congruence θ on a Boolean algebra $\mathbf{A}$, the congruence class $[1]_\theta$ of 1 is a filter of $\mathbf{A}$.

Given a filter F of a Boolean algebra $\mathbf{A}$, θ_F is a congruence, where $(a,b) \in \theta_F$ iff $a \leftrightarrow b \in F$ iff $\{a \to b, b \to a\} \subseteq F$.

Note that a filter is a subset of $\mathbf{A}$ closed under $\{\wedge, \vee, \to, 1\}$ that is *convex* ($x \leq y \leq z$ and $x, z \in F$ implies $y \in F$).

Congruences R, M

Congruences on rings correspond to ideals.

Congruences on ℓ-groups correspond to convex ℓ-subgroups.

Congruences on monoids do not correspond to any particular kind of subset.

Do congruences on residuated lattices correspond to certain subsets?

Congruences and sets

Let $\mathbf{A}$ be a residuated lattice and $a, x \in A$. We define the conjugates $\lambda_a(x) = [a\backslash(xa)] \wedge 1$ and $\rho_a(x) = ax/a \wedge 1$.

An *iterated conjugate* is a composition $\gamma_{a_1}(\gamma_{a_2}(\dots\gamma_{a_n}(x)))$, where $n \in \omega$, $a_1, a_2, \dots, a_n \in A$ and $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$, for all i.

$X \subseteq A$ is called *normal*, if it is closed under conjugates.

We will be considering correspondences between:

- Congruences on $\mathbf{A}$
- Convex, normal subalgebras (CNSs) of $\mathbf{A}$
- Convex , normal (in $\mathbf{A}$) submonoids (CNMs) of $\mathbf{A}^- = \downarrow 1$
- Deductive filters of $\mathbf{A}$: $F \subseteq A$
  - $\uparrow 1 \subseteq F$
  - $a, a\backslash b \in F$ implies $b \in F$ (eqv. $\uparrow F = F$)
  - $a \in F$ implies $a \wedge 1 \in F$ (eqv. F is $\wedge$-closed)
  - $a \in F$ implies $b\backslash ab, ba/b \in F$

Correspondence

If S is a CNS of $\mathbf{A}$, M a SNM of $\mathbf{A}^-$, θ a congruence on $\mathbf{A}$ and F a DF of $\mathbf{A}$, then

1. $M_s(S) = S^-$, $M_c(\theta) = [1]_\theta^-$ and $M_f(F) = F^-$ are SNMs of $\mathbf{A}^-$,
2. $S_m(M) = \Xi(M)$, $S_c(\theta) = [1]_\theta$ and $S_f(F) = \Xi(F^-)$ are CNSs of $\mathbf{A}$,
3. $F_s(S) = \uparrow S$, $F_m(M) = \uparrow M$, and $F_c(\theta) = \uparrow [1]_\theta$ are DFs of $\mathbf{A}$.
4. $\Theta_s(S) = \{(a,b) | a \leftrightarrow b \in S\}$, $\Theta_m(M) = \{(a,b) | a \leftrightarrow b \in M\}$ and $\Theta_f(F) = \{(a,b) | a \leftrightarrow b \in F\} = \{(a,b) | a\backslash b, b\backslash a \in F\}$ are congruences of $\mathbf{A}$.

$a \leftrightarrow b = a\backslash b \wedge b\backslash a \wedge 1$

$\Xi(X) = \{a \in A : x \leq a \leq x\backslash 1$, for some $x \in X\}$.

CNM to CNS

$\Xi(M) = \{a \in A | x \leq a \leq x\backslash 1$, for some $x \in M\}$ is a CNS.

Claim: $a \in \Xi(M)$ iff $\exists y, z \in M$ such that $y \leq a \leq z\backslash 1$.
Indeed, $yz \leq y \leq a \leq z\backslash 1 \leq yz\backslash 1$ and $yz \in M$.

Convexity: If $a, b \in \Xi(M)$, then $\exists x, y \in M$ such that
$x \leq a \leq x\backslash 1$ and $y \leq b \leq y\backslash 1$.
If $a \leq c \leq b$, then $x \leq a \leq c \leq b \leq y\backslash 1$, so $c \in \Xi(M)$.

Subalg.: $xy \leq x \wedge y \leq a \wedge b \leq x\backslash 1 \wedge y\backslash 1 = (x \vee y)\backslash 1 \leq x\backslash 1$

$$x \leq x \vee y \leq a \vee b \leq x\backslash 1 \vee y\backslash 1 \leq (x \wedge y)\backslash 1 \leq (xy)\backslash 1$$

$$xy \leq ab \leq (x\backslash 1)(y\backslash 1) \leq x\backslash(y\backslash 1) = (yx)\backslash 1$$

$$\lambda_a(yx) \leq a\backslash yxa \leq a\backslash[y/(x\backslash 1)]a \leq a\backslash[b/a]a \leq a\backslash b \leq x\backslash(y\backslash 1) = yx\backslash 1$$

$$xy \leq x/(y\backslash 1) \leq a/b \leq (x\backslash 1)/y \leq [x\rho_{(x\backslash 1)/y}(y)]\backslash 1$$

(for $u = (x\backslash 1)/y$ we have $x\rho_u(y)u \leq x\{uy/u\}u \leq xuy \leq 1$)

Normality: As $\lambda_c(x)\lambda_c(x\backslash 1) \leq c\backslash x(x\backslash 1)c \wedge 1 \leq c\backslash c \wedge 1 = 1$,

$$\lambda_c(x) \leq \lambda_c(a) \leq \lambda_c(x\backslash 1) \leq \lambda_c(x)\backslash 1$$

CNS to congruence

$\Theta_s(S) = \{(a,b) | a \leftrightarrow b \in S\}$ is a congruence.
$a \leftrightarrow b = a \backslash b \wedge b \backslash a \wedge 1$

Equivalance: $\Theta_s(S)$ is reflexive and symmetric. If $a \leftrightarrow b, b \leftrightarrow c \in S$, we have

$$(a \leftrightarrow b)(b \leftrightarrow c) \wedge (b \leftrightarrow c)(a \leftrightarrow b) \leq$$

$$\leq (a \backslash b)(b \backslash c) \wedge (c \backslash b)(b \backslash a) \wedge 1 \leq (a \leftrightarrow c) \leq 1.$$

Comptibility: Assume $a \leftrightarrow b \in S$ and $c \in A$.

$a \backslash b \leq ca \backslash cb$ implies $a \leftrightarrow b \leq ca \leftrightarrow cb \leq 1$

$$\lambda_c(a \leftrightarrow b) \leq c \backslash (a \backslash b) c \wedge c \backslash (b \backslash a) c \wedge e \leq ac \leftrightarrow bc \leq 1$$

$(a \wedge c) \cdot (a \leftrightarrow b) \leq a(a \leftrightarrow b) \wedge c(a \leftrightarrow b) \leq b \wedge c$ implies $a \leftrightarrow b \leq (a \wedge c) \backslash (b \wedge c)$. Likewise, $a \leftrightarrow b \leq (b \wedge c) \backslash (a \wedge c)$. So, $a \leftrightarrow b \leq (a \wedge c) \leftrightarrow (b \wedge c) \leq 1$

$a \backslash b \leq (c \backslash a) \backslash (c \backslash b)$ and $b \backslash a \leq (c \backslash b) \backslash (c \backslash a)$ imply $a \leftrightarrow b \leq (c \backslash a) \leftrightarrow (c \backslash b) \leq 1$

CNS to congruence

$$a\backslash b \leq (a\backslash c)/(b\backslash c) \text{ and } b\backslash a \leq (b\backslash c)/(a\backslash c) \text{ imply}$$
$$a \leftrightarrow b \leq (a\backslash c){\leftrightarrow}'(b\backslash c) \leq 1$$

where $a \leftrightarrow' b = a/b \wedge b/a \wedge 1$.

So, $(a\backslash c) \leftrightarrow'(b\backslash c) \in S$ and $(a\backslash c) \leftrightarrow(b\backslash c) \in S$.

Claim: $a \leftrightarrow' b \in S$ iff $a \leftrightarrow b \in S$.

$$\lambda_b(a{\leftrightarrow}'b) = b\backslash[a/b \wedge b/a \wedge 1]b \wedge 1 \leq b\backslash a \wedge 1$$

$$\lambda_b(a{\leftrightarrow}'b) \wedge \lambda_a(a{\leftrightarrow}'b) \leq a \leftrightarrow b \leq 1$$

Lattice isomorphism

1. The CNSs of $\mathbf{A}$, the CNMs of $\mathbf{A}^-$ and the DF of $\mathbf{A}$ form lattices, denoted by $\mathbf{CNS}(\mathbf{A})$, $\mathbf{CNM}(\mathbf{A})$ and $\mathbf{Fil}(\mathbf{A})$, respectively.
2. All the above lattices are isomorphic to the congruence lattice $\mathbf{Con}(\mathbf{A})$ of $\mathbf{A}$ via the maps defined above.
3. The composition of the above maps gives the corresponding map; e.g., $M_s(S_c(\theta)) = M_c(\theta)$.

Claim: S_c and Θ_s are inverse maps.
$S = [1]_{\Theta_s(S)}$: $a \in S$ implies $a \leftrightarrow 1 = a\backslash 1 \wedge a \wedge 1 \in S$.
Conversely, $(a \leftrightarrow 1) \leq a \leq (a \leftrightarrow 1)\backslash 1$.

$\theta = \Theta_s(S_c(\theta))$: If $(a, b) \in \Theta_s([1]_\theta)$, then $a \leftrightarrow b \in [1]_\theta$, so $a \leftrightarrow b \;\theta\; 1$. Therefore, $a \;\theta\; a(a \leftrightarrow b) \leq a(a\backslash b) \leq b$, so $a \vee b \;\theta\; b$. Likewise, $a \vee b \;\theta\; a$, so $a \;\theta\; b$.

Conversely, if $a \;\theta\; b$, then
$1 = (a\backslash a \wedge b\backslash b \wedge 1) \;\theta\; (a\backslash b \wedge b\backslash a \wedge 1) = a \leftrightarrow b$.

Compositions

Claim: $S_f(F) = S_c(\Theta_f(F))$. (Sketch)

If $a \in S_c(\Theta_f(F))$, then $a \; \Theta_f(F) \; 1$, so $a\backslash 1, 1\backslash a \in F$.
Hence $a, 1/a \in F$. Since $1 \in F$, we get $x = a \wedge 1/a \wedge 1 \in F^-$.
Obviously, $x \leq a$; also $a \leq (1/a)\backslash 1 \leq x\backslash 1$.
Thus, $a \in S_f(F)$.

Conversely, if $a \in S_f(F)$, then $x \leq a \leq x\backslash 1$, for some $x \in F^-$.
So, $a \in F$ and $1/(x\backslash 1) \leq 1/a$.
Since, $x \leq 1/(x\backslash 1)$, we have $x \leq 1/a$ and $1/a \in F$.
Thus both $a/1$ and $1/a$ are in F. Hence, $a \in [1]_{\Theta_f(F)}$.

Generation

If X is a subset of A^- and Y is a subset of A, then

1. the CNM $M(X)$ of A^- generated by X is equal to $\Xi^-\Pi\Gamma(X)$.
2. The CNS $S(Y)$ of $\mathbf{A}$ generated by Y is equal to $\Xi\Pi\Gamma\Delta(Y)$.
3. The DF $F(Y)$ of $\mathbf{A}$ generated by $Y \subseteq A$ is equal to $\uparrow \Pi\Gamma(Y) = \uparrow \Pi\Gamma(Y \wedge 1)$.
4. The congruence $\Theta(P)$ on $\mathbf{A}$ generated by $P \subseteq A^2$ is equal to $\Theta_m(M(P'))$, where $P' = \{a \leftrightarrow b | (a, b) \in P\}$.

$$X \wedge 1 = \{x \wedge 1 : x \in X\}$$
$$\Delta(X) = \{x \leftrightarrow 1 : x \in X\}$$
$$\Pi(X) = \{x_1 x_2 \cdots x_n : n \geq 1, x_i \in X\} \cup \{1\}$$
$$\Gamma(X) = \{\gamma(x) : \gamma \text{ is an iterated conjugate } \}$$
$$\Xi(X) = \{a \in A : x \leq a \leq x\backslash 1, \text{ for some } x \in X\}$$
$$\Xi^-(X) = \{a \in A : x \leq a \leq 1, \text{ for some } x \in X\}$$
$$a \leftrightarrow b = a\backslash b \wedge b\backslash a \wedge 1$$

Generation of CNM

Clearly, if M is a CNM of $\mathbf{A}^-$ that contains X, then it contains $\Gamma(X)$, by normality, $\Pi\Gamma(X)$, since M is closed under product, and $\Xi^-\Pi\Gamma(X)$, since M is convex and contains 1. We will now show that $\Xi^-\Pi\Gamma(X)$ itself is a CNM of A^-; it obviously contains X. It is clearly convex and a submonoid of $\mathbf{A}^-$. To show that it is convex, consider $a \in \Xi^-\Pi\Gamma(X)$ and $u \in A$. There are $x_1, \ldots, x_n \in X$ and iterated conjugates $\gamma_1, \ldots, \gamma_n$ such that $\gamma_1(x_1) \cdots \gamma_n(x_n) \leq a \leq 1$. We have

$$\prod \lambda_u(\gamma_i(x_i)) \leq \lambda_u(\prod \gamma_i(x_i)) \leq \lambda_u(a) \leq 1.$$

Idea for $n = 2$:

$$\lambda_u(a_1)\lambda_u(a_2) = (u\backslash a_1 u \wedge 1)(u\backslash a_2 u \wedge 1) \leq (u\backslash a_1 u)(u\backslash a_2 u) \wedge 1$$

$$\leq u\backslash a_1 u(u\backslash a_2 u) \wedge 1 \leq u\backslash a_1 a_2 u \wedge 1 = \lambda_u(a_1 a_2).$$

Also, $\lambda_u(\gamma_i(x_i)) \in \Gamma(X)$ and $\prod \lambda_u(\gamma_i(x_i)) \in \Pi\Gamma(X)$, so $\lambda_u(a) \in \Xi^-\Pi\Gamma(X)$. Likewise, we have $\rho_u(a) \in \Xi^-\Pi\Gamma(X)$.

Subvariety lattice (atoms)

Size

We view RL as the subvariety of $\mathsf{RL_p}$ axiomatized by $0 = 1$.

The subvariety lattices of HA (Heyting algebras) and Br (Brouwerian algebras) are uncountable, hence so are $\mathbf{\Lambda}(\mathsf{RL_p})$ and $\mathbf{\Lambda}(\mathsf{RL})$.

We will

- determine the size of the set of atoms in $\mathbf{\Lambda}(\mathsf{RL_p})$.
- outline a method for finding axiomatizations of certain varieties
- give a description of joins in $\mathbf{\Lambda}(\mathsf{RL_p})$.

The variety BA of Boolean algebras is generated by the 2-element algebra $\mathbf{2}$. BA $=$ HSP$(\mathbf{2})$$=$ V$(\mathbf{2})$.

H: homomorphic images
S: subalgebras
P: direct products
V $=$ HSP

Proof idea: Use the prime ideal-filter theorem for distributive lattices to show that every Boolean algebra is a subdirect product of copies of $\mathbf{2}$.

Subdirect product: A subalgebra of a product such that all projections are onto.

Clearly, $\mathbf{2}$ is subdirectly irreducible.

Subdirectly irreducible: non-trivial and

- it cannot be written as a subdirect product of a family that does not contain it.
- Alt. its congruence lattice is $\Delta \cup \uparrow \mu$.

BA: an atom

The variety BA is an atom in the lattice of subvarieties of pRL.

pRL is a *congruence distributive* variety (RL's have lattice reducts) so Jònsson's Lemma applies:
Given a class $\mathcal{K} \subseteq \mathsf{RL_p}$, the subdirectly irreducible algebras $\mathsf{V}(\mathcal{K})_{SI}$ in the variety generated by a $\mathcal{K}$ are in $\mathsf{HSP_U}(\mathcal{K})$.

An *ultraproduct* $\mathbf{A} \in \mathsf{P_U}(\mathcal{K})$ is obtained by taking

- a product $\prod_{i \in I} A_i$ of $A_i \in \mathcal{K}$ and then
- a quotient $\prod_{i \in I} A_i / \cong_U$ by an ultrafilter U over I (maximal filter on $\mathcal{P}(U)$):
  for $\bar{a}, \bar{b} \in \prod_{i \in I} A_i$, $\bar{a} \cong_U \bar{b}$ iff $\{i \in I : a_i = b_i\} \in U$.

First order formulas persist under ultraproducts.

Now, $\mathsf{HSP_U}(\mathbf{2}) = \{\mathbf{2}, \mathbf{1}\}$, hence $(\mathsf{V}(\mathbf{2}))_{SI} = \{\mathbf{2}\}$.

Recall that $\mathcal{V} = \mathsf{V}(\mathcal{V}_{SI})$.

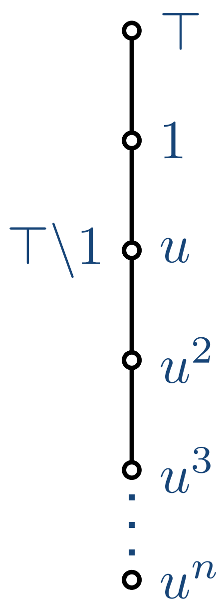
Fin. gen. atoms

We define $\top u = u\top = u$.

Note that $\mathbf{T}_n$ is *strictly simple* (has no non-trivial subalgebras or homomorphic images).

So, $\mathsf{V}(\mathbf{T}_n)$ is an atom of $\mathbf{\Lambda}(\mathsf{RL})$.

Moreover, all these atoms are distinct and $\mathbf{\Lambda}(\mathsf{RL})$ has at least denumerably many atoms.



$\mathbf{T}_n$

Cancellative atoms

Left cancellativity ($ab = ac \Rightarrow b = c$) can be written equationally: $x\backslash(xy) = y$. Right cancellativity is $(yx)/x = y$. CanRL denotes the variety of cancellative RL's.

Prop. There are only 2 cancellative atoms: $\mathsf{V}(\mathbb{Z})$ and $\mathsf{V}(\mathbb{Z}^-)$.

Let $\mathbf{L} \in \mathsf{CanRL}$. For $a \leq 1$, we have $1 \leq 1/a$.

Claim: If $\exists a < 1$ with $1/a = 1$, then $Sg(a) \cong \mathbb{Z}^-$.

Since $a < 1$, we get $a^{n+1} < a^n$, for all $n \in \mathbb{N}$, by order preservation and cancellativity. Moreover, $a^{k+m}/a^m = a^k$ and $a^m/a^{m+k} = 1$, for all $m, k \in \mathbb{N}$.

Claim: If for all $x < 1$, we have $1 < 1/x$, then $\mathbf{L}$ is an ℓ-group.

For $a \in L$ set $x = (1/a)a$. Note that $x \leq 1$, and if $x < 1$, then $1/x = 1/(1/a)a = (1/a)/(1/a) = 1$, cancellativity; so $x = 1$.

The *negative cone* of a RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is the RL $\mathbf{A}^- = (A^-, \wedge, \vee, \cdot, \backslash^{\mathbf{A}^-}, /^{\mathbf{A}^-}, 1)$, where $A^- = \{a \in A : a \leq 1\}$, $a \backslash^{\mathbf{A}^-} b = (a\backslash b) \wedge 1$ and $b /^{\mathbf{A}^-} a = (b/a) \wedge 1$.

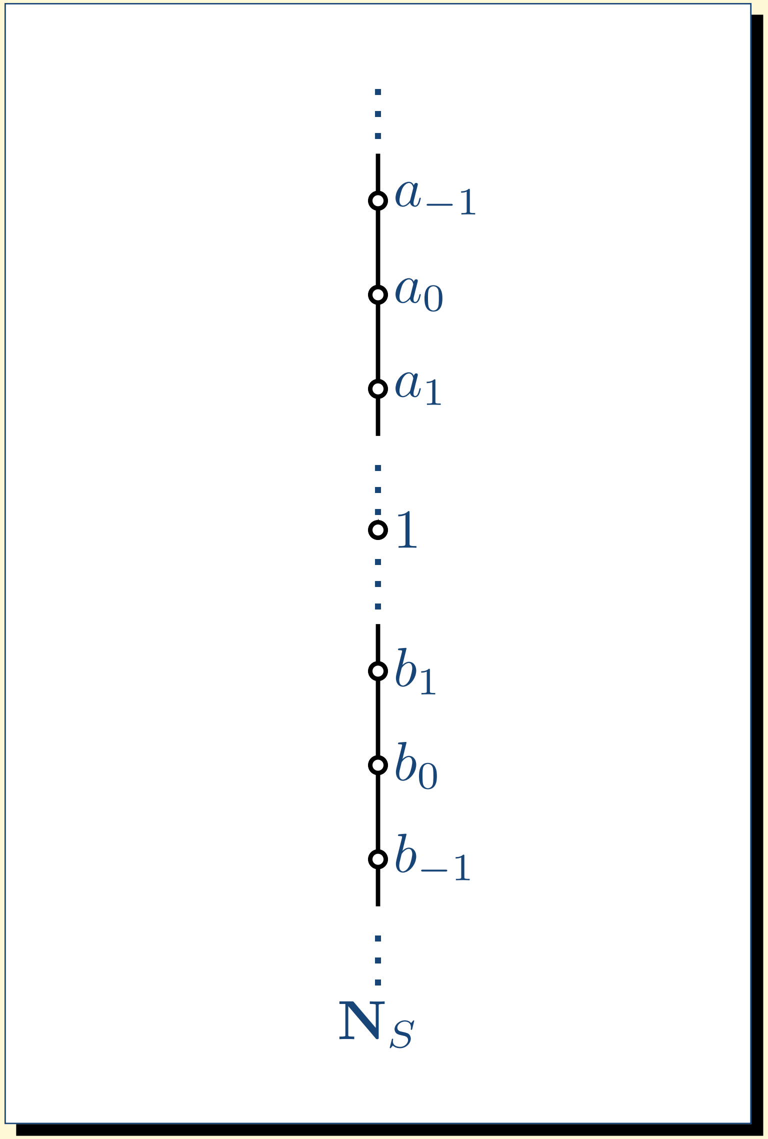
Idempotent rep. atoms

For $S \subseteq \mathbb{Z}$, we define
$a_i b_i = a_i$, if $i \in S$ and
$a_i b_i = b_i$, if $i \notin S$.

Although, we may have

- $S \neq T$, but $\mathbf{N}_S \cong \mathbf{N}_T$
- $\mathbf{N}_S \not\cong \mathbf{N}_T$, but $\mathsf{V}(\mathbf{N}_S) \neq \mathsf{V}(\mathbf{N}_T)$
- $\mathsf{V}(\mathbf{N}_S)$ is not an atom

there are still continuum many atoms $\mathsf{V}(\mathbf{N}_S)$.



Subvariety lattice (joins)

Representable RL's

A residuated lattice is called *representable* (or semi-linear) if it is a subdirect product of totally ordered RL's. RRL denotes the class of representable RL's.

Recall that a totally ordered RL satisfies the first-order formula $(\forall x, y)(x \leq y$ or $y \leq x)$ [$(\forall x, y)(1 \leq x \backslash y$ or $1 \leq y \backslash x)$]

Representable Heyting algebras form a variety axiomatized by $1 = (x \to y) \vee (y \to x)$.

Representable commutative RL's form a variety axiomatized by $1 \leq (x \to y)_{\wedge 1} \vee (y \to x)_{\wedge 1}$.

RRL is a variety axiomatized by $1 \leq \gamma_1(x \backslash y) \vee \gamma_2(y \backslash x)$.

Goal: Given a class $\mathcal{K}$ of RL's axiomatized by a set of positive universal first-order formulas (PUF's), provide an axiomatization for V$(\mathcal{K})$.

Joins

The meet of two varieties in $\mathbf{\Lambda}(\mathsf{RL_p})$ is their intersection.

Also, if $\mathcal{V}_1$ is axiomatized by E_1 and $\mathcal{V}_2$ by E_2, then $\mathcal{V}_1 \wedge \mathcal{V}_2$ is axiomatized by $E_1 \cup E_2$.

On the other hand, the join of two varieties is the variety *generated* by their union.

Also, if $\mathcal{V}_1$ is axiomatized by E_1 and $\mathcal{V}_2$ by E_2, then $\mathcal{V}_1 \vee \mathcal{V}_2$ may not be axiomatized by $E_1 \cap E_2$.

Goals

- Find an axiomatization of $\mathcal{V}_1 \vee \mathcal{V}_2$ in terms of E_1 and E_2.
- Find situations where: if E_1 and E_2 are finite, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is finitely axiomatized.
- Find $\mathcal{V}$ such that its finitely axiomatized subvarieties form a lattice.

Finite basis

If $\mathcal{V}$ is a congruence distributive variety of finite type and $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely axiomatized.

Strictly elementary: Axiomatized by a single FO-sentence.
Finitely SI: Δ is not the intersection of two non-trivial congruences.

Cor. For every variety $\mathcal{V}$ of RL's, if $\mathcal{V}_{FSI}$ is *strictly* elementary, then the finitely axiomatized subvarieties of $\mathcal{V}$ form a lattice.

Pf. For finitely axiomatized subvarieties $\mathcal{V}_1$, $\mathcal{V}_2$, $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI} = (\mathcal{V}_1 \cup \mathcal{V}_2)_{FSI}$ is strictly elementary.

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of RL axiomatized by E_1, E_2, respectively, where E_1, E_2 have *no variables in common*.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the universal closure of (AND E_1) or (AND E_2), over infinitary logic, which is equivalent to the set $\{\forall\forall(\varepsilon_1 \text{ or } \varepsilon_2) : \varepsilon_1 \in E_1, \varepsilon_2 \in E_2\}$ of *positive universal first-order formulas* (PUFs).

In a RL, we say that 1 is *weakly join irreducible*, if for all negative a, b, whenever $1 = \gamma(a) \vee \gamma'(b)$, for all all iterrated conjugates γ, γ', then $a = 1$ or $b = 1$.

Thm. A RL is FSI iff 1 is weakly join-irreducible.

($\Leftarrow$) Let F, G be CNS with $F \cap G = \{1\}$. For all $a \in F^-$ and $b \in G^-$, $1 = \gamma(a) \vee \gamma'(b)$, for all iterated conjugates, because if $\gamma(a), \gamma'(b) \leq u$, then $u \wedge 1 \in F \cap G = \{1\}$, so $1 \leq u$. Since 1 is weakly join-irreducible, $a = 1$ or $b = 1$.

($\Rightarrow$) Let a, b be negative elements and assume that $u \in CNS^-(a) \cap CNS^-(b)$. Then there exist products of iterated conjugates p, q of a, b, resp., such that $p, q \leq u$. If $1 = \gamma(a) \vee \gamma'(b)$, for all iterated conjugates, then $1 = p \vee q$. Thus, $u = 1$ and $CNS^-(a) \cap CNS^-(b) = \{1\}$.
Since $\mathbf{A}$ is FSI, $CNS^-(a) = \{1\}$ or $CNS^-(b) = \{1\}$, hence $a = 1$ of $b = 1$.

PUF's

Every PUF is equivalent to (the universal closure of) a disjunction of conjunctions of equations.

$s = t$ iff $(s \leq t$ and $t \leq s)$ iff $(1 \leq s\backslash t$ and $1 \leq t\backslash s)$.

Every conjunction of equations $1 \leq p_i$ is equivalanent to the equation $1 \leq p_1 \wedge \cdots \wedge p_n$.

So, every PUF is equivalent to a formula of the form

$$\alpha = \forall \overline{x} \ (1 \leq r_1 \text{ or } \cdots \text{ or } 1 \leq r_k)$$

Let $\widetilde{\alpha}_0$ be $(r_1)_{\wedge 1} \ \vee \ \cdots \ \vee \ (r_k)_{\wedge 1} = 1$.

Also, for $m > 0$ and $\aleph_0$ fresh variables Y, we define $\widetilde{\alpha}_m$ as the set of all equations of the form

$$\gamma_1 \ \vee \ \cdots \ \vee \ \gamma_k = 1$$

where $\gamma_i \in \Gamma^m_Y(r_i)$ for each $i \in \{1, \ldots, k\}$. Set $\widetilde{\alpha} = \bigcup_{n \in \omega} \widetilde{\alpha}_n$.

Here $\Gamma^m_Y(a) = \{\pi_{y_1}\pi_{y_2} \cdots \pi_{y_m}(a_{\wedge 1}) \mid y_i \in Y, \pi_{y_i} \in \{\lambda_{y_i}, \rho_{y_i}\}\}$.

PUF and equations

Thm. For a PUF α and a FSI RL $\mathbf{A}$, $\mathbf{A} \models \alpha$ iff $\mathbf{A} \models \widetilde{\alpha}$.

Pf. ($\Rightarrow$) If $\bar{a}$ are elements in A, then $1 \leq r_i(\bar{a})$ for some i.
So, $\gamma(r_i(\bar{a})_{\wedge 1}) = 1$, for all γ; hence,
$\gamma_1(r_1(\bar{a})_{\wedge 1}) \vee \cdots \vee \gamma_k(r_k(\bar{a})_{\wedge 1}) = 1$.

($\Leftarrow$) We have $1 = \gamma_1(r_1(\bar{a})_{\wedge 1}) \vee \cdots \vee \gamma_k(r_k(\bar{a})_{\wedge 1})$, for all γ_i.
Since $\mathbf{A}$ is FSI, 1 is weakly join irreducible, so $r_i(\bar{a})_{\wedge 1} = 1$, for some i; i.e., $r_i(\bar{a}) \leq 1$.

$\alpha = \forall \overline{x}\ (1 \leq r_1$ or $\cdots$ or $1 \leq r_k)$

$\widetilde{\alpha} = \{\gamma_1 \vee \cdots \vee \gamma_k = 1 \mid \gamma_i \in \Gamma_Y(r_i)\}$

Thm. Let $\mathcal{K}$ be a class of RLs axiomatixed by a set Ψ of PUF. Then $\mathsf{V}(\mathcal{K})$ is axiomatized, relative to RL, by $\widetilde{\Psi}$.

Pf. Let $\mathbf{A} \in \mathsf{RL}_{SI}$. By congruence distributivity and Jónsson's Lemma, $\mathbf{A} \in \mathsf{V}(\mathcal{K})$ iff $\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K})$. Furthermore, as PUFs are preserved under H, S and $\mathsf{P_U}$, $\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K})$ iff $\mathbf{A} \in K$. Finally, $\mathbf{A} \in K$ iff $\mathbf{A} \models \Psi$ iff $\mathbf{A} \models \widetilde{\Psi}$.

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of RL axiomatized by E_1, E_2, respectively, where E_1, E_2 have *no variables in common*. The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the set of PUFs $\Psi = \{\forall\forall(1 \leq r_1 \text{ or } 1 \leq r_2) \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2\}$.

Thm. $\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by

$$\widetilde{\Psi} = \{\gamma_1(r_1) \vee \gamma_2(r_2) = 1 \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2, \gamma_i \in \Gamma\}$$

Thm. The variety RRL generated by all totally ordered residuated lattices is axiomatized by the 4-variable identity $\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1$.

Pf. A RL is a chain iff it satisfies $\forall x, y(x \leq y$ or $y \leq x)$, or

$$\forall x, y(1 \leq (x \vee y)\backslash x \text{ or } 1 \leq (x \vee y)\backslash y).$$

Thus, RRL is axiomatized by the identities

$$1 = \gamma_1((x \vee y)\backslash x) \vee \gamma_2((x \vee y)\backslash y);\ \gamma_1, \gamma_2 \in \Gamma \qquad (\Gamma)$$

So, RRL satisfies the identity

$$\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1. \qquad (\lambda, \rho)$$

Conversely, the variety axiomatized by this identity satisfies

$$x \vee y = 1 \Rightarrow \lambda_z(x) \vee y = 1 \qquad x \vee y = 1 \Rightarrow x \vee \rho_w(y) = 1. \quad \text{(imp)}$$

By repeated applications of (imp) on (λ, ρ), we get (Γ).

Finite axiomatization

Let $\beta = \forall x_1 \forall x_2\ (1 \le x_1$ or $1 \le x_2)$ and set $B_m \Rightarrow B_{m+1}\ =$

$$\forall x_1 \forall x_2\, [\,(\forall \overline{y}\, \forall z \text{ AND } \widetilde{\beta}_m) \Longrightarrow (\forall \overline{y}\, \forall z \text{ AND } \widetilde{\beta}_{m+1})\,]$$

Thm. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties of RLs that satisfy $B_m \Rightarrow B_{m+1}$. Then

1. $\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by $\widetilde{\Psi}_m$ + a finite set of equations.
2. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are finitely axiomatized then so is $\mathcal{V}_1 \vee \mathcal{V}_2$

Pf. By congruence distributivity $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI} \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$, so $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI}$ satisfies $B_m \Rightarrow B_{m+1}$. $\mathcal{V}_1 \vee \mathcal{V}_2$ also satisfies $B_m \Rightarrow B_{m+1}$, because the latter is a special Horn sentence (Lyndon) and is preserved under subdirect products.
By compactness of FOL, $B_m \Rightarrow B_{m+1}$ is a consequence of a finite set B of equations, valid in $\mathcal{V}_1 \vee \mathcal{V}_2$.
Note that $\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by $\widetilde{\Psi}$ and, using $B_m \Rightarrow B_{m+1}$, $\widetilde{\Psi}_m$ implies $\widetilde{\Psi}_n$ for all $n > m$.
Hence, $\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by $\widetilde{\Psi}_m \cup B$.

Elementarity

Thm. For any variety $\mathcal{V}$ of RLs, $\mathcal{V}_{FSI}$ is an elementary class iff it satisfies $B_m \Rightarrow B_{m+1}$ for some m.

Cor. For every variety $\mathcal{V}$ of RLs, if $\mathcal{V}_{FSI}$ is elementary, then the finitely axiomatized subvarieties of $\mathcal{V}$ form a lattice.

Applications

RRLs satisfy $B_0 \Rightarrow B_1$.

$x \vee y = 1 \Rightarrow \gamma_1(x) \vee \gamma_2(y) = 1$, for all $\gamma_1, \gamma_2 \in \Gamma^1_Y$.

ℓ-groups satisfy $B_1 \Rightarrow B_2$.

For $a \leq 1$, we have $\lambda_z(\lambda_w(a)) = \lambda_{wz}(a)$ and $\rho_z(a) = \lambda_{z^{-1}}(a)$.

Subcommutative RSs satisfy $B_0 \Rightarrow B_1$.

k-subcommutative RSs are defined by $(x \wedge 1)^k y = y(x \wedge 1)^k$.

Logic

A Hilbert-style axiomatization

(MP) $\{\phi, \phi \to \psi\} \vdash_{\mathbf{HL_e}} \psi$

(B) $\vdash_{\mathbf{HL_e}} (\phi \to \psi) \to [(\psi \to \chi) \to (\phi \to \chi)]$

(C) $\vdash_{\mathbf{HL_e}} [\phi \to (\psi \to \chi)] \to [\psi \to (\phi \to \chi)]$

(I) $\vdash_{\mathbf{HL_e}} \phi \to \phi$

(AD) $\{\phi, \psi\} \vdash_{\mathbf{HL_e}} \phi \wedge \psi$

(CLa) $\vdash_{\mathbf{HL_e}} (\phi \wedge \psi) \to \phi$

(CLb) $\vdash_{\mathbf{HL_e}} (\phi \wedge \psi) \to \psi$

(CR) $\vdash_{\mathbf{HL_e}} [(\phi \to \psi) \wedge (\phi \to \chi)] \to [\phi \to (\psi \wedge \chi)]$

(DRa) $\vdash_{\mathbf{HL_e}} \psi \to (\phi \vee \psi)$

(DRb) $\vdash_{\mathbf{HL_e}} \psi \to (\phi \vee \psi)$

(DL) $\vdash_{\mathbf{HL_e}} ((\phi \to \chi) \wedge (\psi \to \chi)) \to (\phi \vee \psi) \to \chi$

(PR) $\vdash_{\mathbf{HL_e}} \phi \to [\psi \to (\psi \cdot \phi)]$

(PL) $\vdash_{\mathbf{HL_e}} [\psi \to (\phi \to \chi)] \to [(\phi \cdot \psi) \to \chi]$

(U) $\vdash_{\mathbf{HL_e}} 1$

(UP) $\vdash_{\mathbf{HL_e}} 1 \to (\phi \to \phi)$

Substructural logics

The system $\mathbf{HL}$ has the following inference rules:

$$\frac{\phi \quad \phi\backslash\psi}{\psi} \text{ (mp)} \qquad \frac{\phi \quad \psi}{\phi \wedge \psi} \text{ (adj)} \qquad \frac{\phi}{\psi\backslash\phi\psi} \text{ (pn)} \qquad \frac{\phi}{\psi\phi/\psi} \text{ (pn)}$$

We write $\Phi \vdash_{\mathbf{HL}} \psi$, if the formula ψ is provable in $\mathbf{HL}$ from the set of formulas Φ.

We do not allow substitution instances of formulas in Φ.

For example, $p, p\backslash q \nvdash_{\mathbf{HL}} r$.

A set of formulas is called a *substructural logic* if it is closed under $\vdash_{\mathbf{HL}}$ and substitution.

Substructural logics form a lattice $\mathbf{SL}$.

In the following we identify (propositional) formulas over $\{\wedge, \vee, \cdot, \backslash, /, 1\}$ with terms over the same signature.

For a set of equations $E \cup \{s = t\}$, we write

$$E \models_{\mathsf{RL}} s = t$$

if for every residuated lattice $\mathbf{L} \in$ RL and for every homomorphism $f : \mathbf{Fm} \to \mathbf{L}$,

$f(u) = f(v)$, for all $(u = v) \in E$, implies $f(s) = f(t)$.

Theorem. The consequence relation $\vdash_{\mathbf{HL}}$ is *algebraizable*, with RL as an *equivalent algebraic semantics*:

1. if $\Phi \cup \{\psi\}$ is a set of formulas, then
   $\Phi \vdash_{\mathbf{HL}} \psi$ iff $\{1 \leq \phi | \phi \in \Phi\} \models_{\mathsf{RL}} 1 \leq \psi$, and
2. if $E \cup \{t = s\}$ is a set of equations, then
   $E \models_{\mathsf{RL}} t = s$ iff $\{u \backslash v \wedge v \backslash u | (u = v) \in E\} \vdash_{\mathbf{HL}} t \backslash s \wedge s \backslash t$.
3. $s = t \;=\!\!|\models_{\mathsf{RL}} 1 \leq t \backslash s \wedge s \backslash t$
4. $\phi \dashv\vdash_{\mathbf{HL}} 1 \backslash (1 \wedge \phi) \wedge (\phi \wedge 1) \backslash 1$

Theorem. $\mathbf{SL}$ and $\mathbf{\Lambda}(\mathsf{RL})$ are dually isomorphic.

Substructural logics (examples)

Note that $\mathbf{HL}$ does not admit

(C)	$[x \to (y \to z)] \to [y \to (x \to z)]$	$(xy = yx)$
(K)	$y \to (x \to y)$	$(x \leq 1)$
(W)	$[x \to (x \to y)] \to (x \to y)$	$(x \leq x^2)$

Examples of substructural logics include

- classical: (C)+(K)+(W)+ $\neg\neg\phi = \phi$ (DN)
- intuitionistic (Brouwer, Heyting): (C)+(K)+(W)
- many-valued (Łukasiewicz): (C)+(K)+ $(\phi \to \psi) \to \psi = \phi \vee \psi$
- basic (Hajek): (C)+(K)+ $\phi(\phi \to \psi) = \phi \wedge \psi$
- MTL (Esteva, Godo): (C)+(K)+ $(\phi \to \psi) \vee (\psi \to \phi)$
- relevance (Anderson, Belnap): (C)+(W)+ Distrib. (+ DN)
- (MA)linear logic (Girard): (C)

Substructural logics (examples)

Relevance logic deals with relevance.

$p \to (q \to q)$ is not a theorem.
The algebraic models do not satisfy integrality $x \leq 1$.

$p \to (\neg p \to q)$ [or $(p \cdot \neg p) \to q$] is not a theorem, where $\neg p = p \to 0$. The algebraic models do not satisfy $0 \leq x$.

Commutativity and distributivity are OK, so we get *involutive* $\mathcal{CDRL}$ (they satisfy $\neg\neg x = x$).

Intuitionistic logic deals with provability or constructibility.
The algebraic models are Heyting algebras.

Many-valued logic allows different degrees of truth.
$[(p \wedge q) \to r] \leftrightarrow [p \to (q \to r)]$ is not a theorem.
The algebraic models do not satisfy $x \wedge y = x \cdot y$.

Linear logic is resourse sensitive. $p \to (p \to p)$ [or $(p \cdot p) \to p$] and $p \to (p \cdot p)$ are not theorems.
The algebraic models do not satisfy contraction $x \leq x^2$.

The deduction theorem for CPL states:

$$\Sigma, \psi \vdash_{CPL} \phi \quad \text{iff} \quad \Sigma \vdash_{CPL} \psi \to \phi$$

Theorem. Let $\Sigma \cup \Psi \cup \{\phi\} \subseteq Fm_{\mathcal{L}}$ and $\mathbf{L}$ be a logic.

- If $\mathbf{L}$ is commutative, integral and contractive, then
  $$\Sigma, \Psi \vdash_{\mathbf{L}} \phi \quad \text{iff} \quad \Sigma \vdash_{\mathbf{L}} (\bigwedge_{i=1}^{n} \psi_i) \to \phi,$$
  for some $n \in \omega$,and $\psi_i \in \Psi$, $i < n$.
- If $\mathbf{L}$ is commutative and integral, then
  $$\Sigma, \Psi \vdash_{\mathbf{L}} \phi \quad \text{iff} \quad \Sigma \vdash_{\mathbf{L}} (\prod_{i=1}^{n} \psi_i) \to \phi,$$
  for some $n \in \omega$,and $\psi_i \in \Psi$, $i < n$.
- If $\mathbf{L}$ is commutative, then
  $$\Sigma, \Psi \vdash_{\mathbf{L}} \phi \quad \text{iff} \quad \Sigma \vdash_{\mathbf{L}} (\prod_{i=1}^{n}(\psi_i \wedge 1)) \to \phi,$$
  for some $n \in \omega$,and $\psi_i \in \Psi$, $i < n$.
- If $\mathbf{L}$ is any substructural logic, then
  $$\Sigma, \Psi \vdash_{\mathbf{L}} \phi \quad \text{iff} \quad \Sigma \vdash_{\mathbf{L}} (\prod_{i=1}^{n} \gamma_i(\psi_i))\backslash\phi,$$
  for some $n \in \omega$, iterated conjugates γ_i and $\psi_i \in \Psi$, $i < n$.

Applications to logic

- Hilbert systems (Algebraization)
- PLDT (Congruence generation for RL's)
- Maximal consistent logics (Atoms in $\mathbf{\Lambda}(\mathsf{RL})$)
- Axiomatizing intersections of logics (Joins in $\mathbf{\Lambda}(\mathsf{RL})$)
- Translations (Glivenko, Kolmogorov) between logics, e.g., $\vdash_{CPL} \phi$ iff $\vdash_{Int} \neg\neg\phi$ (Structure of $\mathbf{\Lambda}(\mathsf{RL})$ and nuclei)

Algebra	$\leftrightarrow$	Logic
congruence generation	$\leftrightarrow$	PLDT
congruence extension	$\leftrightarrow$	localDT
EDPC	$\leftrightarrow$	deduction theorem
subreduct axiomatization	$\leftrightarrow$	strong seperation (Hilbert)
decid. equational th.	$\leftrightarrow$	decid. provability (Gentzen)
finite generation	$\leftrightarrow$	cut elimination (+ fin. proof)
amalgamation	$\leftrightarrow$	interpolation

Representation - Frames

Lattice frames

A *lattice frame* is a structure $\mathbf{W} = (W, W', N)$ where W and W' are sets and N is a binary relation from W to W'.

If $\mathbf{L}$ is a lattice, $\mathbf{W_L} = (L, L, \leq)$ is a lattice frame.

For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^{\triangleright} = \{b \in W' : x \; N \; b, \text{ for all } x \in X\}$$
$$Y^{\triangleleft} = \{a \in W : a \; N \; y, \text{ for all } y \in Y\}$$

The maps $^{\triangleright} : \mathcal{P}(W) \to \mathcal{P}(W')$ and $^{\triangleleft} : \mathcal{P}(W') \to \mathcal{P}(W)$ form a Galois connection. The map $\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W)$, where $\gamma_N(X) = X^{\triangleright\triangleleft}$, is a closure operator.

Lemma. If $\mathbf{L} = (L, \wedge, \vee)$ is a lattice and γ is a cl.op. on $\mathbf{L}$, then $(\gamma[L], \wedge, \vee_\gamma)$ is a lattice. [$x \vee_\gamma y = \gamma(x \vee y)$.]

Corollary. If $\mathbf{W}$ is a lattice frame then the *Galois algebra* $\mathbf{W}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N})$ is a complete lattice.

If $\mathbf{L}$ is a lattice, $\mathbf{W}_{\mathbf{L}}^+$ is the Dedekind-MacNeille completion of $\mathbf{L}$ and $x \mapsto \{x\}^{\triangleleft}$ is an embedding.

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1)$ where W and W' are sets $N \subseteq W \times W'$, $(W, \circ, 1)$ is a monoid and for all $x, y \in W$ and $w \in W'$ there exist subsets $x \mathbin{\backslash\!\!\backslash} w, w \mathbin{/\!\!/} y \subseteq W'$ such that

$$(x \circ y) \; N \; w \;\Leftrightarrow\; y \; N \; (x \mathbin{\backslash\!\!\backslash} w) \;\Leftrightarrow\; x \; N \; (w \mathbin{/\!\!/} y)$$

If $\mathbf{L}$ is a RL, $\mathbf{W_L} = (L, L, \leq, \cdot, \{1\})$ is a residuated frame.

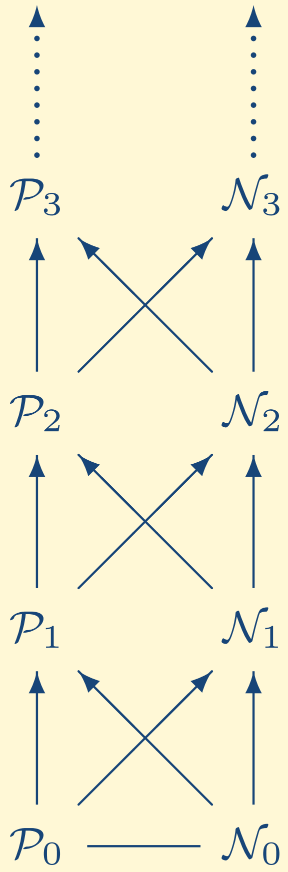
A *nucleus* γ on a residuated lattice $\mathbf{L}$ is a closure operator on L such that $\gamma(x)\gamma(y) \leq \gamma(xy)$ (or $\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$).

Theorem. Given a RL $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ and a nucleus on $\mathbf{L}$, the algebra $\mathbf{L}_\gamma = (L_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, is a residuated lattice, where $x \cdot_\gamma y = \gamma(x \cdot y)$, $x \vee_\gamma y = \gamma(x \vee y)$.

Theorem. If $\mathbf{W}$ is a frame, then γ_N is a nucleus on $\mathcal{P}(W, \circ, \{1\})$.

Corollary. If $\mathbf{W}$ is a residuated frame then the *Galois algebra* $\mathbf{W}^+ = \mathcal{P}(W, \circ, 1)_{\gamma_N}$ is a residuated lattice. Moreover, for $\mathbf{W_L}$, $x \mapsto \{x\}^{\lhd}$ is an embedding.

Formula hierarchy



- Polarity $\{\vee, \cdot, 1\}$, $\{\wedge, \backslash, /\}$
- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables

(P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$

(P2) $\alpha, \beta \in \mathcal{P}_{n+1} \;\Rightarrow\; \alpha \vee \beta, \alpha \cdot \beta, 1 \in \mathcal{P}_{n+1}$

(N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$

(N2) $\alpha, \beta \in \mathcal{N}_{n+1} \;\Rightarrow\; \alpha \wedge \beta \in \mathcal{N}_{n+1}$

(N3) $\alpha \in \mathcal{P}_{n+1}, \beta \in \mathcal{N}_{n+1} \;\Rightarrow\; \alpha \backslash \beta, \beta / \alpha \in \mathcal{N}_{n+1}$

- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\bigvee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\bigwedge, \mathcal{P}_{n+1} \backslash, /\mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$

$$p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leq r$$

- Sequent: $a_1, a_2, \ldots, a_n \Rightarrow a_0$
  $(x \Rightarrow a,\ a \in Fm,\ x \in Fm^*)$

FL

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}\ell) \qquad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{y \circ a \circ z \Rightarrow c \quad y \circ b \circ z \Rightarrow c}{y \circ a \vee b \circ z \Rightarrow c}\ (\vee \text{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c}\ (\backslash \text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \text{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c}\ (\cdot \text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \text{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

FL

$$\frac{x \Rightarrow a \quad u[a] \Rightarrow c}{u[x] \Rightarrow c} \text{ (cut)} \qquad \frac{}{a \Rightarrow a} \text{ (Id)}$$

$$\frac{u[a] \Rightarrow c}{u[a \wedge b] \Rightarrow c} \text{ (}\wedge\text{L}\ell\text{)} \qquad \frac{u[b] \Rightarrow c}{u[a \wedge b] \Rightarrow c} \text{ (}\wedge\text{L}r\text{)} \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b} \text{ (}\wedge\text{R)}$$

$$\frac{u[a] \Rightarrow c \quad u[b] \Rightarrow c}{u[a \vee b] \Rightarrow c} \text{ (}\vee\text{L)} \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b} \text{ (}\vee\text{R}\ell\text{)} \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b} \text{ (}\vee\text{R}r\text{)}$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a\backslash b)] \Rightarrow c} \text{ (}\backslash\text{L)} \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b} \text{ (}\backslash\text{R)}$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[(b/a) \circ x] \Rightarrow c} \text{ (/L)} \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a} \text{ (/R)}$$

$$\frac{u[a \circ b] \Rightarrow c}{u[a \cdot b] \Rightarrow c} \text{ (}\cdot\text{L)} \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b} \text{ (}\cdot\text{R)}$$

$$\frac{|u| \Rightarrow a}{u[1] \Rightarrow a} \text{ (1L)} \qquad \frac{}{\varepsilon \Rightarrow 1} \text{ (1R)}$$

Basic substructural logics

If the sequent s is provable in $\mathbf{FL}$ from the set of *sequents* S, we write $S \vdash_{\mathbf{FL}} s$.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \leq yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \leq x^2$$

$$\frac{[u] \Rightarrow c}{u[x] \Rightarrow c}\ (i) \qquad \text{(integrality)} \qquad x \leq 1$$

We write $\mathbf{FL_{ec}}$ for $\mathbf{FL} + (e) + (c)$.

Theorem. The systems $\mathbf{HL}$ and $\mathbf{FL}$ are *equivalent* via the maps $s(\psi) = (\ \Rightarrow \psi)$ and
$\phi(a_1, a_2, \ldots, a_n \Rightarrow a) = a_n \backslash (\ldots (a_2 \backslash (a_1 \backslash a)) \ldots);$

Examples of frames (FL)

Consider the Gentzen system $\mathbf{FL}$ (full Lambek calculus).

We define the frame $\mathbf{W_{FL}}$, where

- $(W, \circ, \varepsilon)$ to be the free monoid over the set Fm of all formulas
- $W' = S_W \times Fm$, where S_W is the set of all *unary linear polynomials* $u[x] = y{\circ}x{\circ}z$ of W, and
- $x\ N\ (u, a)$ iff $\vdash_{\mathbf{FL}} u[x] \Rightarrow a$.

For

$(u, a) \mathbin{/\!\!/} x = \{(u[\_ \circ x], a)\}$ and $x \mathbin{\backslash\!\!\backslash} (u, a) = \{(u[x \circ \_], a)\}$,

we have

$$\begin{aligned} x \circ y N(u, a) \quad & \text{iff } \vdash_{\mathbf{FL}} u[x \circ y] \Rightarrow a \\ & \text{iff } \vdash_{\mathbf{FL}} u[x{\circ}y] \Rightarrow a \\ & \text{iff } xN(u[\_ \circ y], a) \\ & \text{iff } yN(u[x \circ \_], a). \end{aligned}$$

Examples of frames (FEP)

Let $\mathbf{A}$ be a residuated lattice and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$.

We define the frame $\mathbf{W_{A,B}}$, where

- $(W, \cdot, 1)$ to be the submonoid of $\mathbf{A}$ generated by B,
- $W' = S_B \times B$, where S_W is the set of all *unary linear polynomials* $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$, and
- $x \; N \; (u, b)$ by $u[x] \leq_{\mathbf{A}} b$.

For

$$(u, a) \;/\!\!/\; x = \{(u[\_ \cdot x], a)\} \text{ and } x \;\backslash\!\!\backslash\; (u, a) = \{(u[x \cdot \_], a)\},$$

we have

$$\begin{aligned} x \cdot y N (u, a) \quad & \text{iff } u[x \cdot y] \leq a \\ & \text{iff } x N (u[\_ \cdot y], a) \\ & \text{iff } y N (u[x \cdot \_], a). \end{aligned}$$

GN

$$\frac{xNa \quad aNz}{xNz}\ \text{(CUT)} \qquad \frac{}{aNa}\ \text{(Id)}$$

$$\frac{xNa \quad bNz}{x \circ (a\backslash b)Nz}\ (\backslash\text{L}) \qquad \frac{a \circ xNb}{xNa\backslash b}\ (\backslash\text{R})$$

$$\frac{xNa \quad bNz}{(b/a) \circ xNz}\ (/\text{L}) \qquad \frac{x \circ aNb}{xNb/a}\ (/\text{R})$$

$$\frac{a \circ bNz}{a \cdot bNz}\ (\cdot\text{L}) \qquad \frac{xNa \quad yNb}{x \circ yNa \cdot b}\ (\cdot\text{R})$$

$$\frac{aNz}{a \wedge bNz}\ (\wedge\text{L}\ell) \qquad \frac{bNz}{a \wedge bNz}\ (\wedge\text{L}r) \qquad \frac{xNa \quad xNb}{xNa \wedge b}\ (\wedge\text{R})$$

$$\frac{aNz \quad bNz}{a \vee bNz}\ (\vee\text{L}) \qquad \frac{xNa}{xNa \vee b}\ (\vee\text{R}\ell) \qquad \frac{xNb}{xNa \vee b}\ (\vee\text{R}r)$$

$$\frac{\varepsilon Nz}{1Nz}\ (1\text{L}) \qquad \frac{}{\varepsilon N1}\ (1\text{R})$$

Gentzen frames

The following properties hold for $\mathbf{W_L}$, $\mathbf{W_{FL}}$ and $\mathbf{W_{A,B}}$:

1. $\mathbf{W}$ is a residuated frame
2. $\mathbf{B}$ is a (partial) algebra of the same type, ($\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B}$)
3. B generates $(W, \circ, \varepsilon)$ (as a monoid)
4. W' contains a copy of B ($b \leftrightarrow (id, b)$)
5. N satisfies $\mathbf{GN}$, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

A *cut-free Gentzen frame* is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^+, \quad b \mapsto \{b\}^{\triangleleft}$ is a (partial) homomorphism. (Namely, if $a, b \in B$ and $a \bullet b \in B$ ($\bullet$ is a connective) then $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{W}^+} \{b\}^{\triangleleft}$).

Proof

Key Lemma. Let $(\mathbf{W}, \mathbf{B})$ be a Gentzen frame. For all $a, b \in B$, $k, l \in \mathbf{W}^+$ and for every connective $\bullet$, if $a \bullet b \in B$, $a \in X \subseteq \{a\}^{\lhd}$ and $b \in Y \subseteq \{b\}^{\lhd}$, then

1. $a \bullet_{\mathbf{B}} b \in X \bullet_{\mathbf{W}^+} Y \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$ ($1_{\mathbf{B}} \in 1_{\mathbf{W}^+} \subseteq \{1_{\mathbf{B}}\}^{\lhd}$)
2. In particular, $a \bullet_{\mathbf{B}} b \in \{a\}^{\lhd} \bullet_{\mathbf{W}^+} \{b\}^{\lhd} \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$.
3. Furthermore, because of (CUT), we have equality.

Proof Let $\bullet = \vee$. If $x \in X$, then $x \in \{a\}^{\lhd}$; so $x N a$ and $x N a \vee b$, by ($\vee$Rℓ); hence $x \in \{a \vee b\}^{\lhd}$ and $X \subseteq \{a \vee b\}^{\lhd}$. Likewise $Y \subseteq \{a \vee b\}^{\lhd}$, so $X \cup Y \subseteq \{a \vee b\}^{\lhd}$ and $X \vee Y = \gamma(X \cup Y) \subseteq \{a \vee b\}^{\lhd}$.

On the other hand, let $X \vee Y \subseteq \{z\}^{\lhd}$, for some $z \in W$. Then, $a \in X \subseteq X \vee Y \subseteq \{z\}^{\lhd}$, so $a N z$. Similarly, $b N z$, so $a \vee b N z$ by ($\vee$L), hence $a \vee b \in \{z\}^{\lhd}$. Thus, $a \vee b \in X \vee Y$.

We used that every closed set is an intersection of *basic closed sets* $\{z\}^{\lhd}$, for $z \in W$.

Applications of frames

DM-completion

For a residuated lattice $\mathbf{L}$, we associated the Gentzen frame $(\mathbf{W_L}, \mathbf{L})$.

The underlying poset of $\mathbf{W_L^+}$ is the *Dedekind-MacNeille completion* of the underlying poset reduct of $\mathbf{L}$.

Theorem. The map $x \mapsto x^{\lhd}$ is an embedding of $\mathbf{L}$ into $\mathbf{W_L^+}$.

Completeness - Cut elimination

For every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, let $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{W}^+$ be the homomorphism that extends $\bar{f}(p) = \{f(p)\}^{\lhd}$ (p: variable.)

Corollary. If $(\mathbf{W}, \mathbf{B})$ is a cf Gentzen frame, for every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, we have $f(a) \in \bar{f}(a) \subseteq \downarrow f(a)$. If we have (CUT), then $\bar{f}(a) = \downarrow f(a)$.

We define $\mathbf{W_{FL}} \models x \Rightarrow c$ by $f(x) \; N \; f(c)$, for all f.

Theorem. If $\mathbf{W}^+_{\mathbf{FL}} \models x^{\cdot} \leq c$, then $\mathbf{W_{FL}} \models x \Rightarrow c$.
Idea: For $f : \mathbf{Fm} \to \mathbf{B}$, $f(x) \in \bar{f}(x) \subseteq \bar{f}(c) \subseteq \{f(c)\}^{\lhd}$, so $f(x) \; N \; f(c)$.

Corollary. $\mathbf{FL}$ is complete with respect to $\mathbf{W}^+_{\mathbf{FL}}$.

Corollary. The algebra $\mathbf{W}^+_{\mathbf{FL}}$ generates RL.

The frame $\mathbf{W_{FL^f}}$ corresponds to cut-free $\mathbf{FL}$.

Corollary (CE). $\mathbf{FL}$ and $\mathbf{FL^f}$ prove the same sequents.

Corollary. $\mathbf{FL}$ and the equational theory of RL are decidable.

Finite model property

For $\mathbf{W_{FL}}$, given $(x,z) \in W \times W'$ (if $z = (u,c)$, then $u(x) \Rightarrow c$ is a sequent), we define $(x,z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x,z) and is closed upwards with respect to the rules of $\mathbf{FL^f}$. Note that $(x,z)^{\uparrow}$ is finite.

The new frame $\mathbf{W}'$ associated with $N' = N \cup ((y,v)^{\uparrow})^c$ is residuated and Gentzen.
Clearly, $(N')^c$ is finite, so it has a finite domain $Dom((N')^c)$ and codomain $Cod((N')^c)$.
For every $z \notin Cod((N')^c)$, $\{z\}^{\lhd} = W$. So, $\{\{z\}^{\lhd} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So, $\mathbf{W}'^{+}$ is finite.
Moreover, if $u(x) \Rightarrow c$ is not provable in $\mathbf{FL}$, then it is not valid in $\mathbf{W}'^{+}$.

Corollary. The system $\mathbf{FL}$ has the finite model property.

Corollary. The variety of residuated lattices is generated by its finite members.

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Theorem. Every variety of integral RL's axiomatized by equartions over $\{\vee, \cdot, 1\}$ has the FEP.

- $\mathbf{B}$ embeds in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ via $\{\_\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^+$
- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite
- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$

Corollary. These varieties are generated as quasivarieties by their finite members.

Corollary. The corresponding logics have the *strong finite model property*:
if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model, namely there is $\mathbf{D} \in \mathcal{V}$ and a homomorphism $f : \mathbf{Fm} \to \mathbf{D}$, such that $f(\phi) = 1$, for all $\phi \in \Phi$, but $f(\psi) \neq 1$.

Idea: As every element in $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is an intersection of basic elements. So it suffices to prove that there are only finitely many such elements.

Idea: Replace the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$ by one $\mathbf{W}^{\mathbf{M}}_{\mathbf{A},\mathbf{B}}$, where it is easier to work.

Let $\mathbf{M}$ be the free monoid with unit over the set B and $f : M \to W$ the extension of the identity map.

$$M \xrightarrow{f} W \overset{N}{--} W'$$

.

Idea: Express equations over $\{\vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: monoid terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 y \leq xy \vee yx$

$(x_1 \vee x_2)^2 y \leq (x_1 \vee x_2)y \vee y(x_1 \vee x_2)$

$x_1^2 y \vee x_1 x_2 y \vee x_2 x_1 y \vee x_2^2 y \leq x_1 y \vee x_2 y \vee y x_1 \vee y x_2$

$x_1 x_2 y \leq x_1 y \vee x_2 y \vee y x_1 \vee y x_2$

$$\frac{x_1 y \leq v \quad x_2 y \leq v \quad y x_1 \leq v \quad y x_2 \leq v}{x_1 x_2 y \leq v}$$

$$\frac{x_1 \circ y \; N \; z \quad x_2 \circ y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \circ y \; N \; z} \; R(\varepsilon)$$

Equations 2

Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies $\mathsf{R}(\varepsilon)$ iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Corollary If an equation over $\{\vee, \cdot, 1\}$ is valid in $\mathbf{A}$, then it is also valid in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$, for every partial subalgebra $\mathbf{B}$ of $\mathbf{A}$.

Consequently, $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$.

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \ (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *linear* if all variables in t_0 are distinct.

Theorem. Every system obtained from $\mathbf{FL}$ by adding linear rules has the cut elimination property.

A set of rules of the form $R(\varepsilon)$ is called *reducing* if there is a complexity measure that decreases with upward applications of the rules (and the rules of $\mathbf{FL}$).

Theorem. Every system obtained from $\mathbf{FL}$ by adding linear reducing rules is decidable. The subvariety of residuated lattices axiomatized by the corresponding equations has decidable equational theory.

Amalgamation-Interpolation

Given algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$, maps $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ and Gentzen frames $\mathbf{W_B}, \mathbf{W_C}$, we define the frame $\mathbf{W}$ on $B \cup C$, where N is specified by $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}}\ N\ \beta$ iff there exists $\alpha \in A$ such that $\Gamma_{\mathbf{C}}\ N_{\mathbf{C}}\ g(\alpha)$ and $\Gamma_{\mathbf{B}}, f(\alpha)\ N_{\mathbf{B}}\ \beta$.

Theorem. $\mathbf{W}$ is a Gentzen frame. Hence $^{\triangleleft} : \mathbf{B} \cup \mathbf{C} \to \mathbf{W}^+$ is a quasihomomorhism.

Let $\mathbf{D} = \mathbf{W}^+$ and h, k the restrictions of $^{\triangleleft}$ to $\mathbf{B}$ and $\mathbf{C}$.

Corollary. The maps $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ are homomorphisms. Moreover, injections and surjections transfer: If f is injective (surjective), so is h.

Corollary. Commutative RL has the amalgamation property (f, g injective) and the congruence extension property (f injective, g surjective).

Corollary. $\mathbf{FL_e}$ has the Craig interpolation propety and enjoys the Local Deduction Theorem.

Applications

- Cut-elimination (CE) and finite model property (FMP) for $\mathbf{FL}$, (cyclic) $\mathbf{InFL}$. Generation by finite members for RL, InFL
- The finite embeddability property (FEP) for integral RL with $\{\vee, \cdot, 1\}$-axioms.
- The strong separation property for HL
- The above extend to the non-associative case, as well as with the addition of suitable structural rules
- Amalgamation for commutative RL and interpolation for commutative $\mathbf{FL}$
- (Craig) Interpolation, Robinson Property, disjunction property and Maximova variable separation property for $\mathbf{FL_e}$
- Super-amalgamation, Transferable injections, Congruence extension property for commutative RL

Undecidability

(Un)decidability

Theorem. The quasiequational theory of RL is undecidable. (Because we can embed semigroups/monoids.) The same holds for commutative RL.

Theorem. The equational theory of modular RL is undecidable. (By transfering the corresponding result for modular lattices).

Theorem. The equational theory of commutative, distributive RL is decidable.

Word problem (1)

A finitely presented algebra $\mathbf{A} = (X|R)$ (in a class $\mathcal{K}$) has a *solvable word problem* (WP) if there is an algorithm that, given any pair of words over X, decides if they are equal or not.

A class of algebras has *solvable WP* if all finitely presented algebras in it do.

For example, the varieties of semigroups, groups, ℓ-groups, modular lattices have unsolvable WP.

Main result: The variety CDRL of commutative, distributive residuated lattices has unsolvable WP.

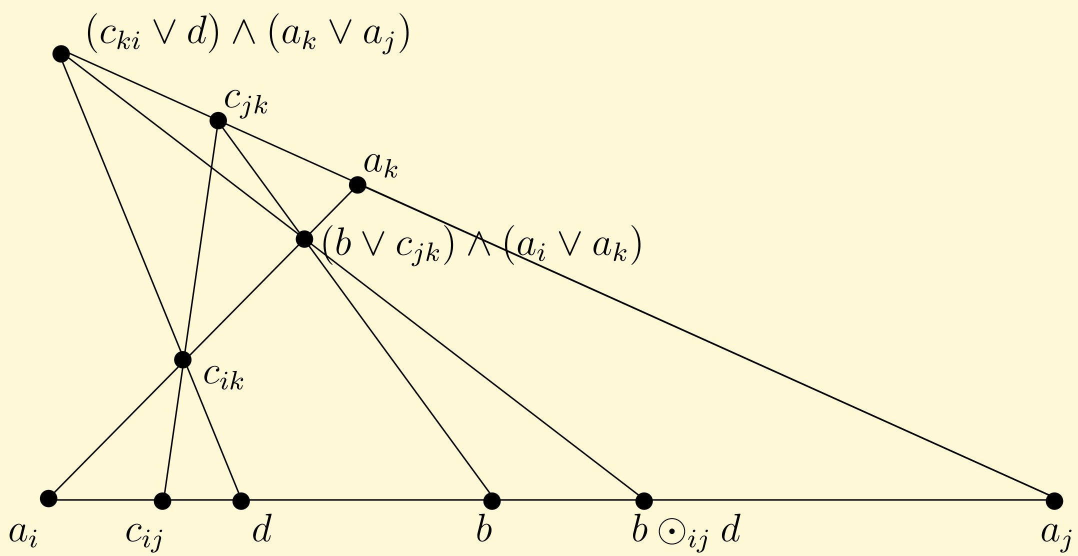
Word problem (2)

Main idea: Embed semigroups, whose WP is unsolvable.

Residuated lattices have a semigroup operation $\cdot$, but commutative semigroups have a decidable WP.

Alternative approach: Come up with another term definable operation $\odot$ in residuated lattices that is associative.

Intuition: Coordinization in projective geometry and modular lattices.



Word problem (3)

We define an n-*frame* in a residuated lattice consisting of elements $a_1, \cdots, a_n$ and c_{ij}, for $1 \leq i < j \leq n$ and satisfying certain conditions (the a_i's are linearly independent, c_{ij} is on the line generated by a_i and a_j etc.).
We use the operations $\vee$ and $\cdot$.

We define the 'line' L_{ij} and the operation $\odot_{ij}$.

Theorem Given an 4-frame in a residuated lattice the algebra $(L_{ij}, \odot_{ij})$ is a semigroup.

Given a finitely presented semigroup $\mathbf{S}$ and a variety $\mathcal{V}$ of residuated lattices, we construct a finitely presented residuated lattice $\mathbf{A}(\mathbf{S}, \mathcal{V})$ in $\mathcal{V}$.

Given a vector space $\mathbf{W}$, its powerset forms a distributive residuated lattice $\mathbf{A_W}$.

Theorem If

1. $\mathcal{V}$ is a variery of distributive residuated lattices containing $\mathbf{A_W}$ for some infinite-dimentional vector space $\mathbf{W}$ and
2. $\mathbf{S}$ is a finitely presented semigroup with unsolvable WP,

then the residuated lattice $\mathbf{A}(\mathbf{S}, \mathcal{V})$ in $\mathcal{V}$ has unsolvable WP.

In the proof we show that for every pair of semigroup words r, s,
S satisfies $r^{\cdot}(\bar{x}) = s^{\cdot}(\bar{x})$ iff $\mathbf{A}(\mathbf{S}, \mathcal{V})$ satisfies $r^{\odot}(\bar{x}') = s^{\odot}(\bar{x}')$.

Corollary The WP of CDRL is unsolvable.

Word problem (5)

A *quasi-equation* is a formula of the form

$$(s_1 = t_1 \;\&\; s_2 = t_2 \;\&\; \cdots \;\&\; s_n = t_n) \Rightarrow s = t$$

The solvability/decidability of the WP states that given any set of equations $s_1 = t_1, s_2 = t_2, \ldots s_n = t_n$ there is an algorithm that decides all quasi-equations of the above form.

The solvability of the *quasi-equational theory* states that there is an algorithm that decides all quasi-equations of the above form.

Corollary The *quasi-equational* theory of CDRL is undecidable.

Corollary The *equational* theory of CDRL is decidable.

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People

Some people involved:

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