Universal Algebra and Computational Complexity
Lecture 1

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Outline

LECTURE 1: Decision problems and Complexity Classes

LECTURE 2: Nondeterminism, Reductions and Complete problems

LECTURE 3: Results and problems from Universal Algebra

Three themes: problems, algorithms, efficiency

A *Decision Problem* is . . .

- A *YES/NO question*
- parametrized by one or more *inputs*.
  - Inputs must:
    - range over an *infinite* class.
    - be "finitistically described"

What we seek:

- An *algorithm* which correctly answers the question for every possible inputs.

What we ask:

- How *efficient* is this algorithm?
- Is there a better (more efficient) algorithm?

Directed Graph Reachability problem (PATH)

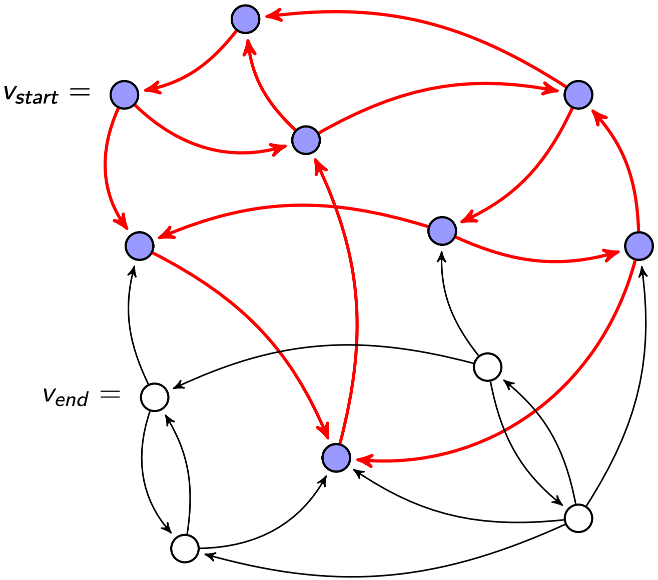
INPUT:

- A finite directed graph $G = (V, E)$
- Two distinguished vertices $v_{start}, v_{end} \in V$.

QUESTION:

- Does there exist in G a *directed path* from v_{start} to v_{end}?
  - (i.e., a sequence $v_{start} = v_0, v_1, v_2, \ldots, v_k = v_{end}$ of vertices such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < k$.)

An Algorithm for PATH



Answer: "NO"

Efficiency of this algorithm

How long does this algorithm take?

- I.e., how many steps?
- ... as a function of the size of the input graph.

I'll give three answers to this.

First answer – Heuristics

Only action is changing a vertex's color.

Only changes possible are

- white $\Rightarrow$ red
- red $\Rightarrow$ blue
- blue $\Rightarrow$ green.

So if $n = |V|$, then the algorithm requires at most $3n$ vertex-color changes.

Second answer – pseudo-code

Assume that $V = \{0, 1, \ldots, n-1\}$ and E is encoded by the adjacency matrix $M_E = [e_{i,j}]$ where

$$e_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{else.} \end{cases}$$

For $i < n$ let c_i be a variable recording the color of vertex i.

Also let *Greenvar* be a variable storing whether there are green-colored vertices.

Second answer – pseudo-code

Algorithm:

- Input n, M_E, *start* and *end*.
- For $i = 0$ to $n-1$ set $c_i := white$.
- Set $c_{start} = green$.
- Set $GreenVar := yes$.
- While $GreenVar = yes$ do:
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$.
  - For $i = 0$ to $n-1$
    - If $c_i = green$ then set $c_i := blue$
  - Set $GreenVar := no$
  - For $i = 0$ to $n-1$
    - If $c_i = red$ then (set $c_i := green$ and set $GreenVar := yes$)
- If $c_{end} = blue$ then output YES; else output NO.

n loops

n^2 cases

$\boxed{\begin{array}{c} O(n^3) \text{ steps} \\ \text{if } n = |V| \end{array}}$

Third answer – machine implementation

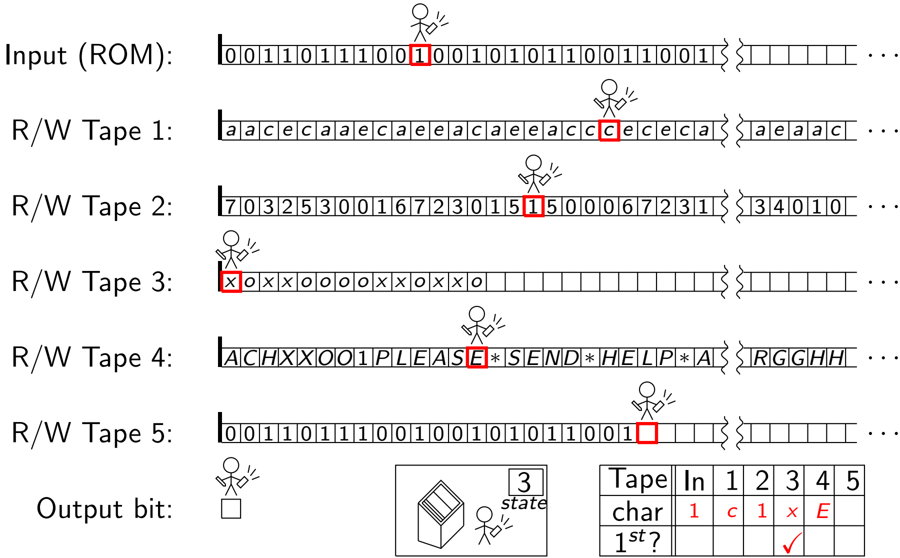
Again assume $V = \{0, 1, \ldots, n-1\}$.

Assume also that $v_{start} = 0$ and $v_{end} = 1$.

Assume the adjacency matrix is presented as a binary string of length n^2.

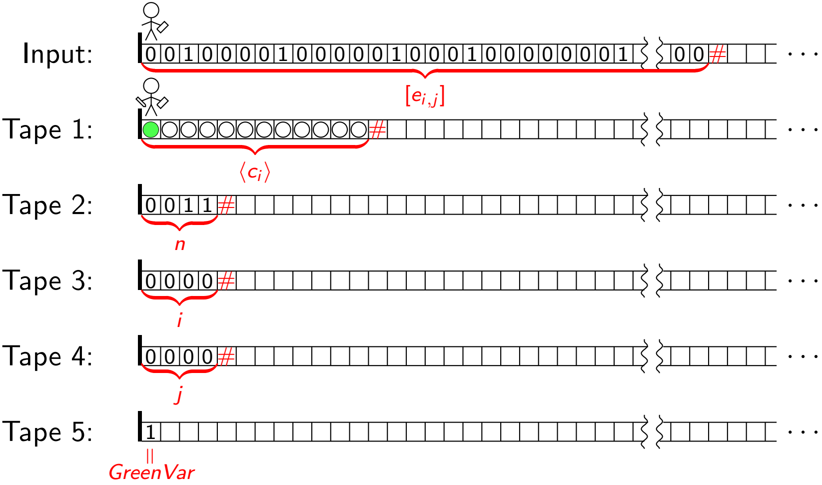
Implement the algorithm on a *Turing machine*.

Turing machine



Tape	In	1	2	3	4	5
char	1	c	1	x	E	
1^{st}?				✓		

Implementing the algorithm for *PATH*



Main loop: For $i, j = 0$ to $n - 1$ ...

Pseudo-code revisited

Point: overhead needed to keep track of i, j, c_i, c_j.

Thus:

- While $GreenVar = yes$ do: — n loops
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$ — n^2 cases
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$. — $O(n \log n)$ steps

SUMMARY: on an input graph $G = (V, E)$ with $|V| = n$, our algorithm decides the answer to $PATH$ using:

Heuristics	$3n$ color changes
Pseudo-code	$O(n^3)$ operations
Turing machine	$O(n^4 \log n)$ steps (Time) $O(n)$ memory cells (Space)

Turing machine complexity

Let $f : \mathbb{N} \to \mathbb{N}$ be given.

A decision problem D (with a specified encoding of its inputs) is:

1. in $TIME(f(N))$ if there exists a Turing machine solving D in at most $O(f(N))$ steps on inputs of length N.
2. in $SPACE(f(N))$ if there exists a Turing machine solving D requiring at most $O(f(N))$ memory cells (not including the input tape) on inputs of length N.

Complexity of *PATH*

Recall that our Turing machine solves *PATH* on graphs with n vertices in

- Time: $O(n^4 \log n)$ steps
- Space: $O(n)$ memory cells.

Since "length N of input" $= n^2$ (when $n = |V|$), we have

$$\begin{aligned} PATH &\in TIME(N^{2+\epsilon}) \\ PATH &\in SPACE(\sqrt{N}) \end{aligned}$$

(Question: can we do better?...)

Another problem: Boolean Formula Value (*FVAL*)

INPUT:

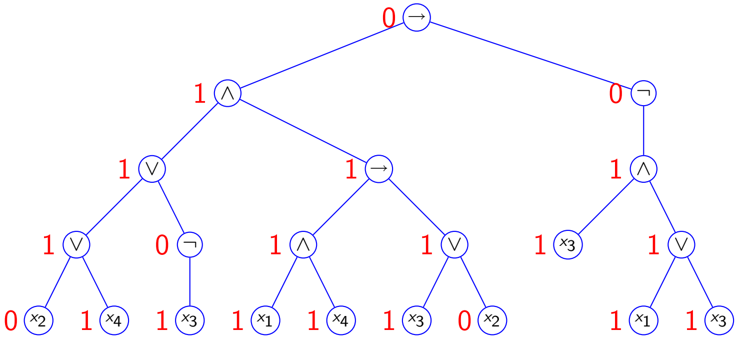
- A boolean formula φ in propositional variables $x_1, \ldots, x_n$.
- A sequence $\mathbf{c} = (c_1, \ldots, c_n) \in \{0, 1\}^n$.

QUESTION:

- Is $\varphi(\mathbf{c}) = 1$?

An algorithm for *FVAL*

$\varphi = ((((x_2 \vee x_4) \vee (\neg(x_3))) \wedge ((x_1 \wedge x_4) \to (x_3 \vee x_2))) \to (\neg(x_3 \wedge (x_1 \vee x_3)))), \quad \mathbf{c} = (1,0,1,1).$



Seems to use *TIME*(N) and *SPACE*(N).
But space can be re-used. In this example, 3 memory bits suffice.

Complexity of *FVAL*

In general, a bottom-up computation, always computing a larger subtree first, can be organized to need only $O(\log|\varphi|)$ intermediate values.

A careful implementation on a Turing machine yields:

Theorem (Nancy Lynch, 1977).

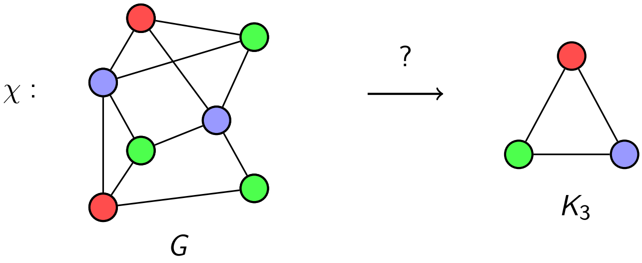
$$\begin{aligned} \mathit{FVAL} &\in \mathit{TIME}(N^{2+\epsilon}) \\ \mathit{FVAL} &\in \mathit{SPACE}(\log N). \end{aligned}$$

A third problem: Graph 3-Colorability ($3COL$)

INPUT: a finite graph $G = (V, E)$.

QUESTION: Is it possible to color the vertices **red**, **green** or **blue**, so that no two adjacent vertices have the same color?

Equivalently: does there exist a homomorphism



An algorithm for $3COL$

Brute force search algorithm:

- For each function $\chi : V \to K_3$:
  - Test if χ works.

$3^{|V|} = 2^{O(\sqrt{N})}$ loops

$O(N^2)$ time,
$O(\sqrt{N})$ space

This at least proves:

$$\begin{aligned} 3COL &\in TIME(2^{O(\sqrt{N})}) \\ 3COL &\in SPACE(\sqrt{N}) \end{aligned}$$

Open problem: can $3COL$ be solved in polynomial time?

A fourth problem: Clone membership (*CLO*)

INPUT:

- A finite algebra $\mathbf{A} = \langle A; f_1, \ldots, f_k \rangle$.
- An operation $g : A^n \to A$.

QUESTION: Is g a term operation of $\mathbf{A}$?

All known algorithms essentially generate the full n-generated free algebra in $V(\mathbf{A})$,

$$\mathbf{F}_n \leq \mathbf{A}^{(A^n)}$$

and test whether $g \in F_n$.

In the worst case this could require as much as $|A^{(A^n)}| = 2^{O(|A|^n)}$ time and space.

Theorem: We cannot solve *CLO* in polynomial time.

Some important complexity classes

1. $P = PTIME = \bigcup_{k=1}^{\infty} TIME(N^k) = TIME(N^{O(1)})$.
2. $PSPACE = \bigcup_{k=1}^{\infty} SPACE(N^k) = SPACE(N^{O(1)})$.

Problems known to be in P are said to be *feasible* or *tractable*.

3. $EXPTIME = \bigcup_{k=1}^{\infty} TIME(2^{N^k}) = TIME(2^{N^{O(1)}})$.
4. $L = LOGSPACE = SPACE(\log(N))$.

$$\begin{array}{ccccccc} L & \subseteq & P & \subseteq & PSPACE & \subseteq & EXPTIME \\ \Cup & & \Cup & & \Cup & & \Cup \\ FVAL & & PATH & & 3COL & & CLO \end{array}$$

Tomorrow

$$
\begin{array}{ccccccc}
L & \subseteq & P & \subseteq & PSPACE & \subseteq & EXPTIME \\
\Cup & & \Cup & & \Cup & & \Cup \\
FVAL & & PATH & & 3COL & & CLO
\end{array}
$$

In tomorrow's lecture I will:

- Introduce "nondeterministic" versions of these 4 classes.
- Introduce problems which are "hardest" for each class.